TOBAM
CORE INVESTMENTS

DIVERSIFICATION DASHBOARD

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Diversification Ratios®

TOBAM's Diversification Ratio® (DR) measures to what extent a portfolio is diversified.
The DR^2 (square of the diversification ratio) measures the number of effective degrees of freedom to which a portfolio is exposed.

As the table shows, the "broad market" indices leave diversification on the table.
In addition to a snapshot of each market's DR^2, the table shows the DR^2 of a well-diversified portfolio (Anti-Benchmark® strategies), and the fraction of available diversification used by the market cap-weighted index.

Universes	DR^2 - Index diversification	DR^2 - Maximum Diversification	% diversification used by index
MSCI All Countries World	4.70	14.39	32.60%
MSCI World	4.52	14.12	32.00%
MSCI Canada	4.93	11.10	44.40%
MSCI US	4.29	10.41	41.20%
MSCI Emerging Markets	3.77	8.72	43.30%
MSCI Pacific Ex-Japan	3.04	7.66	39.70%
MSCI UK	3.50	5.90	59.30%
MSCI Japan	2.32	5.08	45.60%
MSCI EMU	2.46	5.03	48.90%
BofA Merrill Lynch US Corporate & High Yield	5.06	6.65	76.09%
BofA Merrill Lynch Global High Yield	5.63	6.74	83.50%

Source: TOBAM, figures as of September 29, 2017.

On Portfolio Representations

We illustrate in this dashboard some concepts and results from our latest working paper "Portfolio Rho-presentativity" [1]. This is the first in a series, dedicated to this working paper.

Portfolios of financial assets are usually represented by the fraction of their wealth invested in each asset. In this Dashboard, we explore four alternative ways of representing a portfolio, and argue that these representations are useful for at least two reasons.

Firstly, as they provide a picture of the portfolio taken from a different angle, additional portfolio characteristics and insights will come into light. Secondly, these alternative representations may be used to build new portfolios. Indeed, instead of using capital weights as variables for constructing a portfolio, using a different representation allows building portfolios with potentially different and interesting properties.

Alternative Representations considered

Given an investment universe of assets, in this Dashboard we consider the following four alternative portfolio representations of a portfolio w with weights $(w_1, \dots, w_n)$ on assets $A_1, \dots, A_n$.

- *Contribution to Average Volatility:* the weight of each asset is multiplied by its volatility, and all contributions are rescaled in order to sum to one:

$$Contribution\ to\ Avg.Volatility = \frac{w \odot \sigma}{\langle \sigma, w \rangle} \propto w_i \sigma_i$$

where the sign $\propto$ means "proportional to".

In this representation, assets that are more volatile or that have higher weights produce relatively higher contributions. As such, a portfolio comprising of large contributions to its average volatility may be worth monitoring.

- *Contribution to Risk:* the weight of each asset is multiplied by its volatility and by the correlation of the portfolio to the asset. The contributions obtained are then rescaled in order to sum to one:

$$Contribution\ to\ Risk = \frac{1}{\sigma(w)^2}\ w \odot (\sum w)\ \propto\ w_i \sigma_i \varrho(w, A_i).$$

  This representation is similar to the contributions to the average volatility and has the same property outlined above. In addition, assets that have a relatively higher contribution to the Risk of the portfolio will be relatively more correlated to it. As such, these assets are worth monitoring as they may have an oversized contribution to the portfolio absolute performance. Conversely, assets having a negligible contribution to risk could be viewed as having an "effective weight" of zero, despite having a possibly non-zero capital weight.

- The third representation is the one that is explored in our article Portfolio Rho-presentativity*: The Correlation Spectrum* is the correlation of the portfolio to each asset of the universe:

$$Correlation\ Spectrum = \frac{1}{\sigma(w)} \sum w \oslash \boldsymbol{\sigma}\ =\ \rho(w)\ =\ \varrho(w, A_i).$$

  This representation is different from the previous two: it is not systematically equal to zero for assets that do not belong to the portfolio. As with the Contribution to Risk representation, assets that have a high correlation to the portfolio are worth monitoring; note however, that as opposed to the Contribution to Risk, the correlation Spectrum identifies such assets, even if they do not belong to the portfolio.

- *Marginal Risk Contribution:* is the product of the volatility of each asset and the correlation of the portfolio to the asset:

$$Marginal\ Risk\ Contribution = \frac{1}{\sigma(w)} \sum w = \sigma\ \odot \rho(w) = \sigma_i \varrho(w, A_i)$$

  This representation gives the sensitivity of the overall volatility of the portfolio, to a small change in allocation w_i. Note that the sum of the Marginal Contributions is different than one in general. However, its weighted average (using the portfolio weights) sums to its volatility $\sigma(w)$. Finally, we note that dividing each of its components by its corresponding asset volatility gives the Correlation Spectrum.

Portfolios resulting from using alternative representations

A key property of the *Contribution to Average Volatility*, *Correlation Spectrum* and *Marginal Risk Contribution* is that there exists a unique unlevered portfolio having a particular representation (injection). This is also true of the *Contribution to Risk*, when considering long-only portfolios. As such, these alternative representations can be used in order to build new and unique long-only portfolios.

Consider for example the Portfolio Construction Process that seeks to equalize the values taken by a particular representation. The following portfolios are then obtained using different portfolio representations:

- *the Equal Weight (EW)* portfolio, that weights stocks equally;

- *the Equal Volatility Weight (EVW)* portfolio, whose contributions to its weighted average volatility are equal;

- *the Equal Risk Contribution (ERC) long-only* portfolio, for which the contributions to its risk are equal (without a long-only restriction, there are a priori 2^n such portfolios [2]);

- *the Most Diversified Portfolio (MDP)*, whose correlation spectrum has equal values, if possible in a long only set up; and, if not possible, whose correlation spectrum has equal values for each asset that belong to the portfolio, and a higher value otherwise (instead of being equal to zero for assets that do not belong to the portfolio, as with the first two representations). As such, the MDP is not obtained from equalizing its correlations to all assets, but rather by maximizing its minimum correlation to all assets, as shown in [1]. It can otherwise be built by maximizing the Diversification Ratio®.

- *the Minimum Volatility (MV)* is the portfolio whose marginal contributions to its risk are equal for each asset that belong to the portfolio and have a higher value otherwise. As such, the Minimum Variance (MV) portfolio is not obtained by equalizing its marginal contributions to its risk, but can be built by maximizing its minimum marginal volatility contribution, as shown in [1]. It can otherwise be built by minimizing its volatility.

A synthetic illustration

The goal of this section is to show that using alternative representations provides additional insights when analyzing a portfolio. In order to do so, we analyze the portfolios that were introduced in the previous section using a four assets example.

Setup

Our synthetic example features four assets which correlations are positive (similarly to a typical equity universe for example). Their correlations were chosen so that assets (A1,…,A4) have a decreasing average correlation, thanks to the decreasing correlation of A1 to (A2,…,A4) and to the constant correlation of assets (A2,…,A4) with each other. Finally, heterogeneous asset volatilities were chosen such that the differences between the portfolios studied could be illustrated easily.

Table 1. Asset Volatilities and Correlations

Stocks	Volatilities	Correlation				Average Correlation[1]
		A1	A2	A3	A4	
A1	40%	100%	80%	50%	30%	53%
A2	60%	80%	100%	20%	20%	40%
A3	20%	50%	20%	100%	20%	30%
A4	10%	30%	20%	20%	100%	23%

Source: TOBAM. For illustration purpose only. [1]Average correlation to the other three assets.

Portfolios Volatilities & Diversification

Table 2 below provides the volatilities and Diversification Ratio® of the portfolios. As expected, the MV displays the lowest volatility but also the lowest Diversification Ratio®, while the MDP displays the second lowest volatility and the highest Diversification Ratio. Finally, the EVW and ERC have similar volatility and diversification, thanks to their use of the assets volatility as inputs. The EW on the other hand displays the highest volatility and a low Diversification Ratio®.

Table 2. Portfolios Volatilities & Diversification Ratios[2]

	Objectives				
	MDP	EW	EVW	MV	ERC
Portfolio Volatility	12%	27%	15%	10%	14%
Average Volatility	18%	33%	21%	11%	20%
Squared Diversification Ratios® (DR^2)	2.14	1.49	1.90	1.43	1.97

Source: TOBAM. For illustration purpose only.

Portfolio Analysis

- Table 3 below provides the weights of our five portfolios:

Table 3. Portfolio weights

Weights					
Stocks	MDP	EW	EVW	MV	ERC
A1	0%	25%	13%	0%	10%
A2	10%	25%	9%	0%	8%
A3	30%	25%	26%	14%	26%
A4	60%	25%	52%	86%	56%

Stocks		
Volatilities	Average Correlation	Stocks
40%	53%	A1
60%	40%	A2
20%	30%	A3
10%	23%	A4

Source: TOBAM. For illustration purpose only.

The following observations could be made from this Table:

1. The MV is highly concentrated -as measured by weights- in the fourth asset, and it excludes two assets.
2. The MDP has a high 60% weight in the fourth asset, and it excludes the first.
3. Both the EVW and ERC are also somewhat concentrated in the fourth asset.
4. The EW is the only portfolio that is… equally-weighted!

- Table 4 below provides the Contributions to the Average Volatility of the five portfolios:

Table 4. Portfolio Contribution to Average Volatility

Contribution to Average Volatility					
Stocks	MDP	EW	EVW	MV	ERC
A1	0%	31%	25%	0%	20%
A2	33%	46%	25%	0%	24%
A3	33%	15%	25%	25%	27%
A4	33%	8%	25%	75%	28%

Stocks		
Volatilities	Average Correlation	Stocks
40%	53%	A1
60%	40%	A2
20%	30%	A3
10%	23%	A4

Source: TOBAM. For illustration purpose only.

The following observations can be made from this table, *using this particular representation*:

1. The MV becomes marginally less concentrated in the fourth asset (75% vs 86%). This is understandable as this representation takes the volatility of each asset into account.

2. The MDP becomes in fact as "diversified" in the fourth asset as with the two others, as measured by Contribution to Average Volatility.

3. EVW is the one that is… equally-volatility-weighted! And ERC is not far away…

- Tables 5 below provide the Contribution to Risk (CR) and Marginal Contribution to Risk (MCR) of the five portfolios:

Tables 5. Portfolio Contribution to Risk and Marginal Contribution to Risk

Contribution to Risk					
Stocks	MDP	EW	EVW	MV	ERC
A1	0%	36%	31%	0%	25%
A2	33%	52%	26%	0%	25%
A3	33%	9%	23%	14%	25%
A4	33%	3%	20%	86%	25%

Stocks		
Volatilities	Average Correlation	Stocks
40%	53%	A1
60%	40%	A2
20%	30%	A3
10%	23%	A4

Marginal Risk Contribution					
Stocks	MDP	EW	EVW	MV	ERC
A1	31%	38%	36%	17%	35%
A2	41%	55%	46%	14%	44%
A3	14%	10%	13%	10%	13%
A4	7%	4%	6%	10%	6%

Stocks		
Volatilities	Average Correlation	Stocks
40%	53%	A1
60%	40%	A2
20%	30%	A3
10%	23%	A4

Source: TOBAM. For illustration purpose only.

The following observations can be made from this table, keeping Tables 3 and 4 in mind:

1. The MV *Contributions to Risk* are actually the same as its weights shown in Table 3. This is in fact a general property of the MV, since any deviation from it would mean opportunities for volatility reduction. This property can be useful in order to check that an optimizer has found a candidate portfolio with minimum volatility, as it does not involve any optimization.

2. The MV *Marginal Contributions to Risk* are equal for all assets that belong to the portfolio (10%), while being higher otherwise. This property can be shown to be general. It was furthermore proven in [1] that Minimum Variance portfolio is also the portfolio that maximizes its minimum marginal contribution to risk.

3. Reminder (as previously seen in Dashboard10/2015 [3]): The MDP *Contribution to Risk* is the same as its *Contributions to Average Volatility* shown in Table 4, and the MDP is actually the only portfolio satisfying this property.

- Table 6 below illustrates the Correlation Spectrum of the five portfolios:

Table 6. Correlation spectrum

Correlation Spectrum					
Stocks	MDP	EW	EVW	MV	ERC
A1	78%	95%	90%	42%	87%
A2	68%	92%	76%	24%	73%
A3	68%	51%	66%	48%	67%
A4	68%	36%	59%	96%	63%
DR^2	2.14	1.49	1.90	1.43	1.97

Stocks		
Volatilities	Average Correlation	Stocks
40%	53%	A1
60%	40%	A2
20%	30%	A3
10%	23%	A4

Source: TOBAM. For illustration purpose only.

The portfolio "risk drivers" are well identified, the following observations can be made from this Table:

1. The MV is more correlated to the assets it owns (A3, A4), than to the ones it does not own (A1, A2). This is not surprising as the MV has a 96% correlation with the fourth asset. A such, the main driver of this portfolio is (A4).

2. On the opposite, the MDP is so diversified that it is always more correlated to the asset it does not own (A1) than to the other assets, to which it has an equal correlation. Note that this property characterizes the MDP -without having recourse to any optimization- as proved in [4]. Finally, we notice that the main drivers of the MDP are very well diversified, with all assets having a correlation greater than 68% to it.

3. The EW, EVW show similar *Correlation Spectra*, with a maximum value for (A1), the asset with the highest average correlation. However we notice that the main drivers of the EW are (A1,A2) which have a correlation to it in excess of 92%. The drivers of the EVW on the other hand are more diversified, and we notice that its DR^2 is indeed significantly higher.

4. The ERC, has a *Correlation Spectrum* that is close to that of the MDP for the assets that belong to the MDP portfolio, although the fourth asset is less represented in the ERC with a correlation of 63% vs. 68%. Also, the ERC invests in all assets mechanically, which generally limits its opportunities to be highly diversified as measured by the Diversification Ratio®.

Conclusion

In this Dashboard, we explored five ways of representing a portfolio: *simple weights*, and 4 alternatives to it: *Contribution to Average Volatility, Contribution to Risk, Marginal Contribution to Risk* and the *Correlation Spectrum*.

We argued that these alternative representations can be used for building portfolios. Indeed, applying the same portfolio construction process to the previously mentioned portfolio representations leads for example to the EVW, ERC, MV and MDP.

Finally, using a synthetic four assets example, we have shown that these alternative portfolio representations provide useful complements to analyze portfolio weights alone. Note that in a future Dashboard we will provide an analysis of real world portfolio correlation spectra as a complement to the synthetic examples used herein.

References

[1] Froidure, Tristan, Khalid Jalalzai, and Yves Choueifaty. "Portfolio Rho-presentativity." (2017): https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2971867

[2] Spinu, Florin. "An algorithm for computing risk parity weights." (2013).

[3] TOBAM Dashboard. “A Property of the Most Diversified Portfolio: the equality of two different risk contribution measures”. October 2015.

[4] Choueifaty, Yves, Tristan Froidure, and Julien Reynier. "Properties of the most diversified portfolio." (2011).

For more information

TOBAM is an asset management company offering innovative investment capabilities whose aim is to maximize diversification. TOBAM's Maximum Diversification® approach, supported by original, patented research and a mathematical definition of diversification, provides clients with diversified core exposure, in both the equity and fixed income markets. The company manages $8.8 billion (September 2017). Its team includes 49 investment professionals.

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