Toward Maximum Diversification

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Diversification has been at the center of finance for over 50 years. And to paraphrase Markowitz [1952], diversification is the only free lunch in finance. Much effort has gone into developing modern portfolio theory within the Markowitz mean-variance framework. Perhaps foremost amongst those efforts is the capital asset pricing model (CAPM) developed by Sharpe [1964]. While brilliant in its simplicity and clarity, years of examination have led to a vigorous debate about whether the assumptions upon which the model depends reflect real market conditions and whether its conclusion can be transposed to actual portfolio management.

A separate set of arguments concerns the dynamic aspects of portfolio construction. Accounting for dynamic changes in the portfolio has led to an examination of these dynamic changes as a source of return. Fernholz and Shay [1982] stated that constant-proportion portfolios earned additional returns over the returns earned by buy-and-hold portfolios. Booth and Fama [1992] described these additional returns as diversification returns. Although they provide useful insights, the different examinations of multiperiod rebalancing effects are not directly useful for portfolio construction because they only deal with portfolio dynamics after the original weights have been decided.

Although very useful in describing and understanding portfolio construction issues, the mean-variance framework has some practical problems. For example, while variance can be estimated with a fair level of confidence, returns are so much more difficult to estimate that most popular models, such as CAPM and Black-Litterman, have in one way or another completely put them aside. It is now increasingly popular to claim that the market capitalization–weighted indices are not efficient. Several alternative empirical solutions have been suggested, such as fundamental indexation and equal weights.

In this article, we investigate the theoretical and empirical properties of diversification as a criterion in portfolio construction. We compare the behavior of the resulting portfolio to common, simple strategies, such as market cap–weighted indices, minimum-variance portfolios, and equal-weight portfolios.

DEFINITION OF THE DIVERSIFICATION RATIO AND MOST-DIVERSIFIED PORTFOLIO

We begin by mathematically defining how we measure the diversification of a portfolio.

Let $X_1, X_2, \ldots, X_N$ be the risky assets of universe U. For simplification, we will consider X_i to be stocks. Let V be the covariance matrix of these assets and C the correlation matrix.

Let $\Sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ . \\ . \\ \sigma_N \end{bmatrix}$ be the vector of asset volatilities.

Any portfolio P will be noted $P = (w_{p1}, w_{p2}, \ldots, w_{pN})$, with $\Sigma_{i=1}^{N} w_{pi} = 1$.

We define the *diversification ratio* of any portfolio P, denoted $D(P)$, as the following:

$$D(P) = \frac{P' \cdot \Sigma}{\sqrt{P'VP}} \qquad (1)$$

The diversification ratio is the ratio of the weighted average of volatilities divided by the portfolio volatility.

Let Γ be a set of linear constraints applied to the weights of portfolio P. One usual set of constraints is the long-only constraint (i.e., all weights must be positive). The portfolio, which under the set of constraints Γ maximizes the diversification ratio in universe U, is the *Most-Diversified Portfolio*, denoted as $M(\Gamma, U)$.

An intuitive understanding of the way diversification works in portfolio construction can be gained from the following two examples.

Example 1

Suppose we have an investment universe of two stocks, A and B, with a correlation strictly lower than 1, and with respective volatilities of 15% and 30%. In this case, diversification means that we want both stocks to equally contribute to portfolio volatility. Their respective weights in the Most-Diversified Portfolio would thus be 66.6% for stock A and 33.3% for stock B (inversely proportional to volatility).

Example 2

Suppose we have an investment universe of three stocks. Let us assume that two are banking stocks with a high correlation of 0.9 and that the third stock, a pharmaceutical stock, has a correlation of 0.1 with each of the two banking stocks. Suppose, for simplicity, that the volatilities are all equal. The weights of the Most-Diversified Portfolio, according to the result just obtained, are 25.7% for each of the banking stocks and 48.6% for the pharmaceutical stock.

THEORETICAL RESULTS

The diversification ratio of any long-only portfolio will be strictly higher than 1 except when the portfolio is equivalent to a mono-asset portfolio, in which case the diversification ratio will be equal to 1.

If the expected excess returns of assets are proportional to their risks (volatilities), then $ER(P) = kP'\Sigma$, where k is a constant, and maximizing $D(P)$ is equivalent to maximizing $\frac{ER(P)}{\sqrt{P'VP}}$, which is the Sharpe ratio of the portfolio. In this case, the Most-Diversified Portfolio is also the tangency portfolio.

To provide a better understanding of this ratio, and also to simplify the math, we transpose the problem to a synthetic universe in which all the stocks have the same expected volatility.

Suppose that investors can lend and borrow cash at the same rate. We can then define the synthetic assets Y_1, $Y_2, \ldots, Y_N$ by

$$Y_i = \frac{X_i}{\sigma_i} + \left(1 - \frac{1}{\sigma_i}\right)\$$$

where \$ is the risk-free asset. We now have the universe, U_S, of the following assets $Y_1, Y_2, \ldots, Y_N$. In this universe,

the volatility σ_{Si} of Y_i is equal to 1, and $\Sigma_S = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}$

In $D(S) = \frac{S'\Sigma_S}{\sqrt{S'V_S S}}$, S is a portfolio composed of the synthetic assets, and V_S is the covariance matrix of the synthetic assets. If we have $S'\Sigma_S = 1$, then maximizing $D(S)$ is equivalent to maximizing $\frac{1}{\sqrt{S'V_S S}}$ under constraints Γ_S. Because all Y_i have a normalized volatility of 1 and because correlation does not change with leverage, V_S is equal to the correlation matrix C of our initial assets, so that maximizing the diversification ratio is equivalent to minimizing

$$S'CS \tag{2}$$

Thus, in a universe in which all stocks have the same volatility, we minimize the variance, which is indeed the benefit we expect from diversification.

When building a real portfolio, we need to reconstruct synthetic assets by holding real assets plus (or minus) some cash. If $S = (w_{S1}, w_{S2}, \ldots, w_{SN})$ denotes the optimal weights for the synthetic assets, then the optimal portfolio M of real assets will be

$$M = \left(\frac{w_{S1}}{\sigma_1}, \frac{w_{S2}}{\sigma_2}, \ldots, \frac{w_{SN}}{\sigma_N}, \left(1 - \sum_{i=1}^{N} \frac{w_{Si}}{\sigma_i} \right) \$ \right)$$

PROPERTIES

If C is invertible and $\Gamma = \varnothing$, then $S = M(\Gamma_S, U_S)$ is unique, and we have the following analytical results:

$$S \propto C^{-1}1 \tag{3}$$

The synthetic asset weights, S, are proportional to the inverse of the correlation matrix C times 1, a vector of ones the same size as the number of assets.

Once again, we can transform the synthetic assets back to the portfolio of original assets by dividing each synthetic portfolio weight by the volatility of that asset and rescaling the portfolio to be 100% invested. If we denote the vector of weights of the original assets as M, then we can write

$$M \propto \sigma^{-1}C^{-1}1 \tag{4}$$

where σ is a diagonal matrix of the asset volatilities.

Now, consider the properties of asset correlation in the context of the Most-Diversified Portfolio. In a similar manner, we can calculate the correlation of an arbitrary portfolio P with the Most-Diversified Portfolio M. Because M is inversely proportional to σ and C, we can write

$$M = \kappa\sigma^{-1}C^{-1}1$$

where κ is a constant factor.

Thus,

$$\rho_{P,M} = \frac{P'\sigma C\sigma M}{\sigma_P\sigma_M} = \frac{P'\sigma C\sigma\kappa\sigma^{-1}C^{-1}1}{\sigma_P\sigma_M} = \frac{\sum_i w_i\sigma_i}{\sigma_P}\frac{\kappa}{\sigma_M} = D(P)\frac{\kappa}{\sigma_M} \tag{5}$$

This means that the correlation of portfolio P with the Most-Diversified Portfolio M is proportional to the diversification ratio of portfolio P, namely $D(P)$.

Now, consider the correlations between single stocks and the Most-Diversified Portfolio. The diversification ratio of a single stock is 1, because there is no diversification. Using Formula (5), we calculate the correlation of asset i, which has a weight vector w_i, whose ith asset's weight is 1 and other weights are 0, with the vector of the Most-Diversified Portfolio holdings M. We obtain

$$\rho_{i,M} = \frac{\kappa}{\sigma_M} \tag{6}$$

Remarkably, the correlation of asset i with the Most-Diversified Portfolio is the same for every one of the assets. Thus, we can identify the Most-Diversified Portfolio as being the one in which all assets have the same positive correlation to it.

The special case that P is the Most-Diversified Portfolio M in Formula (5) leads us to the result that

$$\kappa = \frac{\sigma_M}{D(M)} \tag{7}$$

so that we have identified the constant κ. Thus, we can rewrite the correlation between a general portfolio P and the Most-Diversified Portfolio M as the ratio of their diversification ratios, as follows:

$$\rho_{P,M} = \frac{D(P)}{D(M)} \tag{8}$$

with this information we can obtain a single (diversification) factor model which resembles the CAPM in form, but now identifies the correlation as the ratio of the diversification levels,

$$R_P = \alpha_P + \frac{\sigma_P}{\sigma_M}\frac{D(P)}{D(M)} R_M + \varepsilon_P \quad (9)$$

where R represents an excess return over cash, and α_P and ε_P are the constant and error terms, respectively, normally associated with regression.

Long-Only Portfolios

In the real world, Γ is usually not empty, and includes the constraint of having positive weights. As such, the properties we described for the unconstrained problem are still true for the subuniverse of securities that is composed of the stocks selected by the constrained Most-Diversified Portfolio.

The subsequent results focus on the long-only Most-Diversified Portfolio. Two consequences of this are that 1) the positivity constraint will reduce the potential impact of estimation errors, and 2) being long-only ensures that the portfolio will have a positive exposure to the equity risk premium.

Thus, all non-zero-weighted assets have the identical correlation to the Most-Diversified Portfolio. Zero-weighted assets, excluded from the Most-Diversified Portfolio in the optimization, have correlations to the Most-Diversified Portfolio that are higher than the non-zero-weighted assets in the Most-Diversified Portfolio. This is consistent with identifying the Most-Diversified Portfolio subject to the constraints applied.

Other Properties

If all of the stocks in the universe have the same volatility, then the Most-Diversified Portfolio is equal to the *global minimum-variance portfolio*. Furthermore, if we continuously rebalance the Most-Diversified Portfolio, and because it is a market cap–independent methodology, the Most-Diversified Portfolio should get a significant part of the benefits from diversification returns when compared to a pure buy-and-hold strategy (Booth and Fama [1992]).

EMPIRICAL RESULTS

In this section, we explain the methodology used in our analysis and the results for Eurozone and U.S. equities. Additionally, we discuss the biases in the methodology and present an analysis of the performance results. We conclude the section with a review of the issues relating to stock selection and the uniqueness of the optimal portfolio.

Methodology

A number of steps need to be addressed before testing the Most-Diversified Portfolio. First, a universe of assets must be selected, and the returns data for these assets must be collected to cover at least a full market cycle. Care should be taken to establish that the data are accurate, particularly in regard to splits, dividends, and, most significantly, survivorship bias.

Given clean returns data, the covariance matrix must next be estimated. Because this is the full information set used to construct the portfolio, it is important to examine the impact of estimation errors on the resulting portfolio. A variety of ways exists to estimate covariances, such as simple windows, decayed weighting, GARCH, and Bayesian update methodologies. Although estimation errors often occur at the levels of volatility and correlation, the hierarchies of correlations are more stable. Indeed, we find that portfolios built using differently estimated covariance matrices have similar characteristics. Changing the frequency of data and the estimation period has little impact on the final results. Even portfolios built on forward-looking covariance matrices (having perfect covariance foresight) have only slightly different results than when using historical covariances.

We must also be aware that optimizers tend to allocate more risk to factors whose volatility has been underestimated (see Michaud [1998]). This is especially true for long–short portfolios built from very large universes where multicollinearity is likely. A simple way to address this issue is to add positivity constraints to the optimization program. In the case of multicollinearity, the fact that the optimal portfolio might not be unique is not a real problem for the portfolio manager—it just provides more choice, as in choosing between equivalent long-only portfolios. It is possible to further limit the impact of estimation errors by adding upper weight limits to the program.

For purposes of comparison, we maximize the diversification ratio, defined in Equation (1), at every month-end for different universes of securities. We compare the results for the long-only Most-Diversified Portfolio with the market cap–weighted benchmark, minimum-variance portfolio, and equal-weight portfolio. We analyze two different regional equity markets, the U.S. and the Eurozone.

We use Standard & Poor's (S&P) 500 Index data from December 1990 to February 2008 as the daily performance series for U.S. equities, and the Dow Jones (DJ) Euro Stoxx Large Cap Index data from December 1990 to February 2008 as the daily performance series for Eurozone equities.

The covariance matrix is computed using 250 days of daily returns. The starting date for the empirical test is December 1991. For computational reasons, we exclude from the universe, for month-end computations, all stocks having less than a 250-day price history.

Because the portfolios are long-only, all weights must be positive. We also limit the contribution to risk to 4% per asset. In order to conform to an asset managers' framework, we also constrain the month-end weights to comply with UCITS III rules (i.e., the maximum weight per security is 10%, and the sum of weights above 5% must be lower than 40%).

Results for Eurozone and U.S. Equities

The results of the empirical tests for the Most-Diversified Portfolio are summarized in Exhibits 1 and 2. The Most-Diversified Portfolio consistently delivers superior risk-adjusted returns in both regions. As expected, it is consistently less risky than the market cap–weighted indices (i.e., volatility is 13.9% versus 17.9% for Eurozone equities, and 12.7% versus 13.4% for U.S. equities). The Most-Diversified Portfolio shows a higher Sharpe ratio than the market cap–weighted benchmark, minimum-variance portfolio, and equal-weight portfolio over the entire period.

In order to further analyze the behavior during different market conditions, we split the backtest results into two subperiods:

Subperiod 1—1992 to 2000 (i.e., the end of the dot-com bubble)

Subperiod 2—2001 to 2008

Biases and Analysis of Performance

It is clear that all market cap–independent methodologies tend to be less biased toward large capitalizations than market cap–weighted indices. Therefore, a comparison of the Most-Diversified Portfolio to market capitalization–weighted indices should always show a size bias. Other biases (relative to market capitalization–weighted indices) can appear, even as the inverse of the index's bias.

To measure the importance of factor bias in the empirical results, we performed a three-factor Fama–French [1993, 1996] regression of the performance of the portfolios versus the market, HML (high-minus-low book value), and SMB (small-minus-big capitalization) factors. The results are shown in Exhibits 3 and 4.

Exhibit 3 shows that for the full period the intercept is significantly positive for the most-diversified Eurozone portfolio, with an annualized excess return (intercept) of 6.0% and a *t*-stat of 4.14. These figures compare to 5.1% and 3.51, respectively, for the minimum-variance portfolio, and 0.6% and 1.16, respectively, for the equal-weight portfolio. The hierarchy of results is confirmed over the two subperiods.

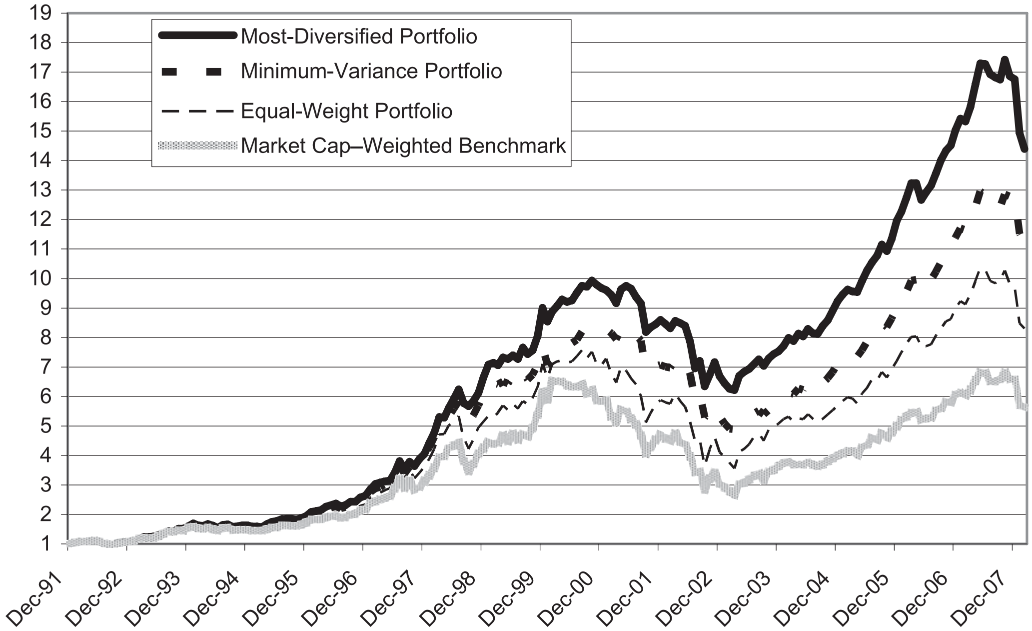
Exhibit 4 shows results of the same nature for the most-diversified U.S. portfolio, with an annualized excess return (intercept) of 3.1% and a *t*-stat of 1.83. These figures compare to 2.2% and 1.40, respectively, for the minimum-variance portfolio, and 1.2% and 2.27, respectively, for the equal-weight portfolio.

More broadly, we analyzed the active returns of the most-diversified Eurozone portfolio with the Lehman Brothers Equity Risk Analysis (ERA) factor model for the period 1999–2008 (i.e., the period of availability for the factor model). The results, shown in Exhibit 5, indicate that the dominant factor explaining the outperformance over the period is specific risk, meaning that about 18% (out of 48%) of the outperformance cannot be explained by the predefined factors of the model. This performance arises from real stock-specific risk, omitted common risk factors, and changes in exposure to factors.

Stock Selection Issues and Uniqueness of the Optimal Portfolio

If the correlation matrix is not invertible, the solution may not be unique. But because all possible portfolios bring maximum diversification, we are indifferent to the solution. We see that, in empirical tests, running the Most-Diversified Portfolio model on three subuniverses (obtained by randomly excluding one-third of the universe each time) gives very similar results.

EXHIBIT 1
Comparison of Eurozone Equity Portfolios, 1992–2008



Full Period 1992–2008	MDP	B	MV	EW
Annualized Return	17.9%	11.3%	16.1%	14.0%
Annualized Excess Return	13.3%	6.6%	11.5%	9.4%
Annualized Volatility	13.9%	17.9%	13.3%	18.1%
Sharpe Ratio	0.96	0.37	0.87	0.52
1992–2000	MDP	B	MV	EW
Annualized Return	28.7%	21.6%	26.5%	24.2%
Annualized Excess Return	22.9%	15.8%	20.7%	18.4%
Annualized Volatility	13.8%	16.8%	13.0%	16.4%
Sharpe Ratio	1.66	0.94	1.59	1.12
2001–2008	MDP	B	MV	EW
Annualized Return	5.7%	–0.5%	4.3%	2.3%
Annualized Excess Return	2.6%	–3.7%	1.2%	–0.8%
Annualized Volatility	13.4%	18.9%	13.1%	19.6%
Sharpe Ratio	0.21	–0.20	0.10	–0.06

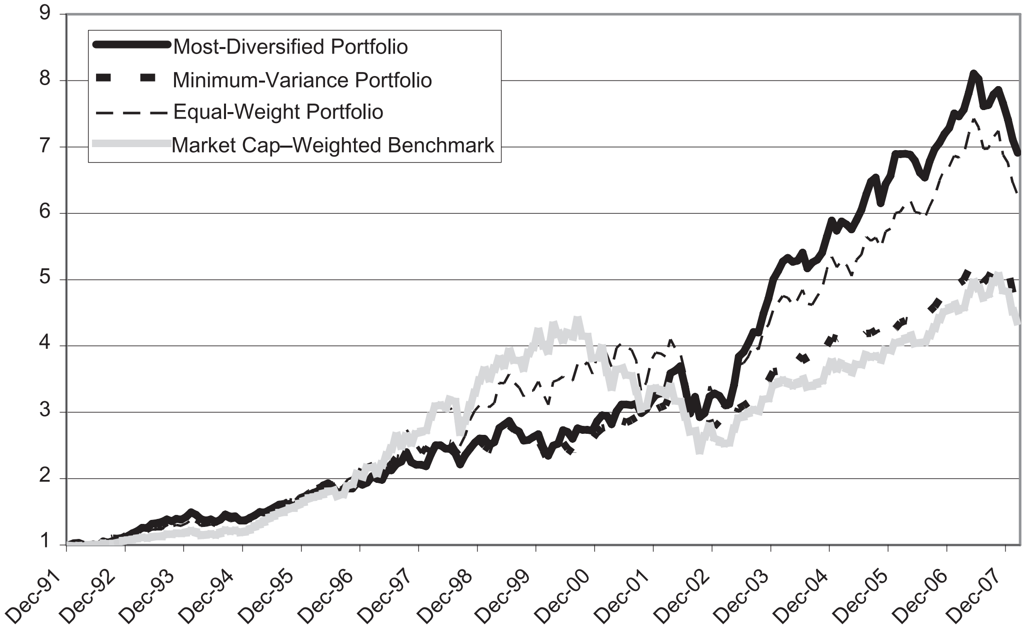
Note: MDP is the Most-Diversified Portfolio. B represents the market cap–weighted benchmark, the Dow Jones EuroStoxx Large Cap Total Return Index. MV represents a minimum-variance portfolio, and EW represents an equally weighted portfolio, both on the same universe as the benchmark B. The chart shows total return indices.

Exhibit 6 shows the performance of the Most-Diversified Eurozone Portfolio and its three subsets versus its benchmark. The same test on the U.S. universe produces similar results. These results show that the Most-Diversified Portfolios tend to allocate risk to risk factors much more than to specific stocks or sectors, even though the average number of stocks in a Most-Diversified Portfolio is relatively low (between 30 and 60).

Diversification Ratio

Exhibit 7 shows the changes in diversification ratios through time for the Most-Diversified Eurozone Portfolio and the Eurozone benchmark. We can see that although the levels of diversification vary through time as a result of the changes in the levels of correlation, the

EXHIBIT 2
Comparison of U.S. Equity Portfolios, 1992–2008



Full Period 1992–2008	MDP	B	MV	EW
Annualized Return	12.7%	9.6%	10.1%	12.0%
Annualized Excess Return	8.4%	5.2%	5.8%	7.7%
Annualized Volatility	12.7%	13.4%	9.9%	14.3%
Sharpe Ratio	0.66	0.39	0.59	0.54
1992–2000	MDP	B	MV	EW
Annualized Return	12.4%	16.1%	11.6%	16.1%
Annualized Excess Return	7.3%	10.9%	6.5%	10.9%
Annualized Volatility	11.9%	13.1%	10.6%	13.1%
Sharpe Ratio	0.61	0.83	0.61	0.84
2001–2008	MDP	B	MV	EW
Annualized Return	13.0%	1.9%	8.3%	7.1%
Annualized Excess Return	9.8%	–1.4%	5.0%	3.8%
Annualized Volatility	13.8%	13.4%	9.1%	15.6%
Sharpe Ratio	0.69	–0.14	0.53	0.21

Note: MDP is the Most-Diversified Portfolio. B represents the market cap–weighted benchmark, the Standard & Poor's (S&P) 500 Total Return Index. MV represents a minimum-variance portfolio, and EW represents an equally weighted portfolio, both on the same universe as the benchmark B. The exhibit shows total return indices.

Most-Diversified Portfolio is always about 1.5 times as diversified as the benchmark.

CONDITIONS FOR OPTIMALITY

Let us consider a world in which investors can borrow and lend money at the same risk-free rate. We will assume that the investor's objective utility function is to maximize the Sharpe ratio of their portfolio of risky assets, before leveraging or deleveraging it with cash. What kind of expected returns would imply the optimality (in terms of Sharpe ratio) of the different strategies?

EXHIBIT 3

Fama-French Monthly Regression Coefficients, Eurozone Equities, 1993–2008

FULL PERIOD	MKT (Xs)	HML	SMB	Intercept
Most-Diversified Portfolio	0.71	0.03	0.21	0.50%
(tstat)	(31.37)	(0.75)	(4.63)	(4.14)
Minimum-Variance Portfolio	0.68	0.14	0.13	0.43%
(tstat)	(29.46)	(3.23)	(2.81)	(3.51)
Equal-Weight Portfolio	1.01	0.11	0.24	0.05%
(tstat)	(115.85)	(6.48)	(14.05)	(1.16)
1992–2000	**MKT (Xs)**	**HML**	**SMB**	**Intercept**
Most-Diversified Portfolio	0.81	−0.06	0.42	0.37%
(tstat)	(25.11)	(−1.15)	(6.87)	(2.25)
Minimum-Variance Portfolio	0.76	0.08	0.31	0.40%
(tstat)	(23.71)	(1.51)	(5.24)	(2.47)
Equal-Weight Portfolio	1.01	0.11	0.23	0.03%
(tstat)	(64.88)	(4.31)	(7.86)	(0.40)
2001–2008	**MKT (Xs)**	**HML**	**SMB**	**Intercept**
Most-Diversified Portfolio	0.64	0.22	0.10	0.25%
(tstat)	(22.16)	(3.20)	(1.50)	(1.56)
Minimum-Variance Portfolio	0.62	0.28	0.00	0.14%
(tstat)	(19.55)	(3.65)	(−0.05)	(0.78)
Equal-Weight Portfolio	1.00	0.11	0.27	0.07%
(tstat)	(111.09)	(5.15)	(13.49)	(1.38)

Monthly data are used. MKT is the benchmark's excess return over one-month LIBOR EUR; HML is the difference in monthly performance between Dow Jones *Euro Stoxx Large Cap Value and Growth Indices; SMB is the difference in monthly performance between the smallest 30% and the biggest 30% of stocks in the index (in terms of weights); and Intercept is a monthly excess return.*

Most-Diversified Portfolio

Recall that the Most-Diversified Portfolio is optimal if the stocks' expected returns are proportional to their volatilities; that is, $E(R_i) = k\sigma_i$, where k is a constant factor and σ_i is the volatility of stock i.

Market Cap–Weighted Benchmark

Considering the CAPM assumptions, a market cap–weighted benchmark will be optimal, and we can state

$$E(R_i) = \beta_i E(R_B) = \rho_{i,B} \frac{\sigma_i}{\sigma_B} E(R_B) \tag{10}$$

where $E(R_i)$ is the expected excess return of stock i, $E(R_B)$ is the expected excess return of the benchmark (a proxy for the market portfolio), and $\rho_{i,B}$ is the correlation between stock i and the benchmark. For simplification we assume the risk-free rate is 0%.

If we consider $E(R_B)$ and σ_B to be given for the period considered, we have

$$E(R_i) = K\rho_{i,B}\sigma_i \tag{11}$$

EXHIBIT 4
Fama-French Monthly Regression Coefficients, U.S. Equities, 1993–2008

FULL PERIOD	MKT (Xs)	HML	SMB	Intercept
Most-Diversified Portfolio	0.64	0.18	0.44	0.26%
(tstat)	(16.86)	(3.01)	(8.95)	(1.83)
Minimum-Variance Portfolio	0.51	0.30	0.15	0.19%
(tstat)	(14.30)	(5.19)	(3.20)	(1.40)
Equal-Weight Portfolio	0.95	0.20	0.37	0.10%
(tstat)	(82.32)	(10.91)	(25.14)	(2.27)
1992–2000	MKT (Xs)	HML	SMB	Intercept
Most-Diversified Portfolio	0.67	0.11	0.53	0.08%
(tstat)	(13.94)	(1.57)	(7.71)	(0.45)
Minimum-Variance Portfolio	0.60	0.29	0.28	0.06%
(tstat)	(12.70)	(4.43)	(4.23)	(0.33)
Equal-Weight Portfolio	0.96	0.22	0.38	0.10%
(tstat)	(63.45)	(10.41)	(17.69)	(1.67)
2001–2008	MKT (Xs)	HML	SMB	Intercept
Most-Diversified Portfolio	0.65	0.36	0.34	0.50%
(tstat)	(9.89)	(2.85)	(4.45)	(2.19)
Minimum-Variance Portfolio	0.46	0.19	0.07	0.35%
(tstat)	(8.12)	(1.80)	(1.07)	(1.78)
Equal-Weight Portfolio	0.94	0.12	0.38	0.10%
(tstat)	(49.41)	(3.38)	(16.86)	(1.53)

Monthly data are used. MKT is the benchmark's excess return over one-month LIBOR USD; HML is the difference in monthly performance between the S&P 500 Value and Growth Indices; SMB is the difference in monthly performance between the smallest 30% and the biggest 30% of stocks in the index (in terms of weights); and Intercept is a monthly excess return.

where K is a constant. In other words, the stocks' expected returns that are implied by the optimality of the market cap–weighted benchmark are proportional to their total risk (volatility) and their correlation to the benchmark.

Minimum-Variance Portfolio

In this case, the expected returns that make the minimum-variance portfolio optimal are equal for all assets,

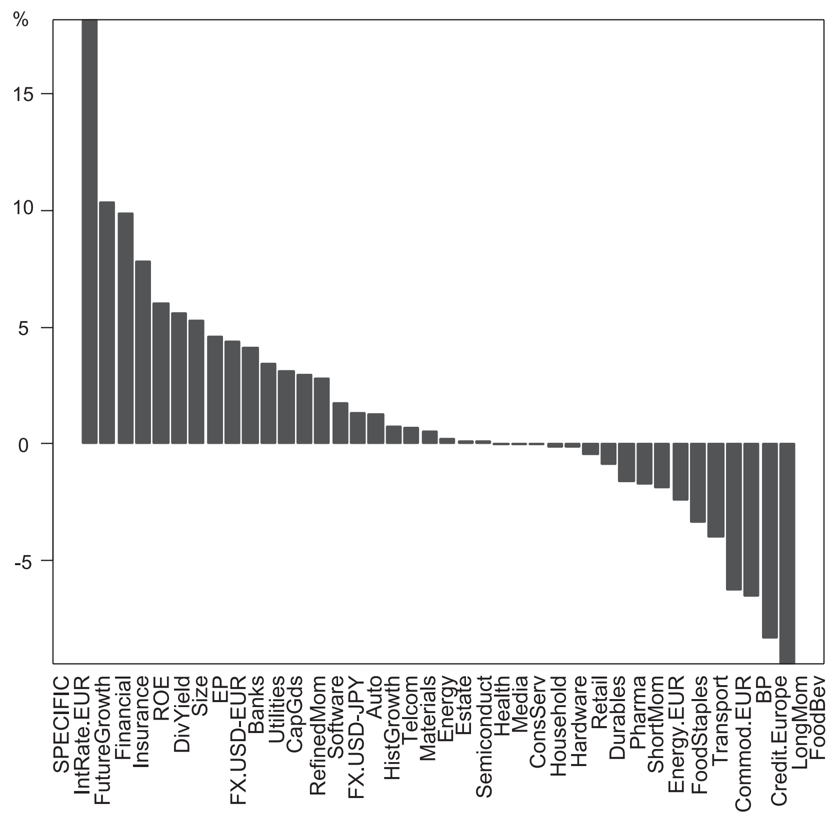
$$E(R_i) = K \tag{12}$$

Economic Interpretation of the Most-Diversified Portfolio

What assumptions would explain that the Most-Diversified Portfolio is better (in terms of Sharpe ratio) than the market cap–weighted benchmark? We can see that, although the Most-Diversified Portfolio is ultimately very different from the benchmark, the implied expected returns from the Most-Diversified Portfolio are not very different from those of the market cap–weighted benchmark. Actually, the only difference resides in the "correct pricing" of individual assets' correlations to the benchmark. In other words, we need correlations (to the

EXHIBIT 5

Active Returns Factor Attribution for Eurozone Most-Diversified Portfolio, April 1999–February 2008



SPECIFIC	0.182
IntRate.EUR	0.104
FutureGrowth	0.099
Financial	0.078
Insurance	0.060
ROE	0.056
DivYield	0.053
Size	0.046
EP	0.044
FX.USD-EUR	0.041
Banks	0.035
Utilities	0.032
CapGds	0.030
RefinedMom	0.028
Software	0.018
FX.USD-JPY	0.014
Auto	0.013
HistGrowth	0.008
Telcom	0.007
Materials	0.005
Energy	0.003
Estate	0.001
Semiconduct	0.001
Health	−0.000
Media	−0.000
ConsServ	−0.000
Household	−0.001
Hardware	−0.001
Retail	−0.005
Durables	−0.009
Pharma	−0.016
ShortMom	−0.017
Energy.EUR	−0.019
FoodStaples	−0.024
Transport	−0.034
Commod.EUR	−0.040
BP	−0.062
Credit.Europe	−0.065
LongMom	−0.083
FoodBev	−0.094
TOTAL	0.486

Note: The factor model used is the Lehman Brothers Equity Risk Analysis (ERA) model. The returns are computed by cumulating monthly active returns, which is a different process from taking the difference between cumulative portfolio retrurns and cumulative benchmark returns.

market) to be only partially taken into account by the market when securities' prices are determined. Why would this be the case in the real world?

The following assumptions are consistent with a market environment that could explain the dominance of the Most-Diversified Portfolio over market-cap indices:

- Investors are rational (i.e., all else being equal, if a security has a higher volatility, investors expect a higher return).
- The market has enough efficiency to prevent arbitrage opportunities at the single-stock level (i.e., security prices reflect all public information; in other words, securities are correctly priced on a stand-alone basis).
- Forecasts of volatilities are accurate.
- All other forecasts are either inaccurate or not taken into account in the pricing of securities.

EXHIBIT 6

Empirical Performance of Eurozone Most-Diversified Portfolios, Full Universe and Three Subsets, 1992–2008

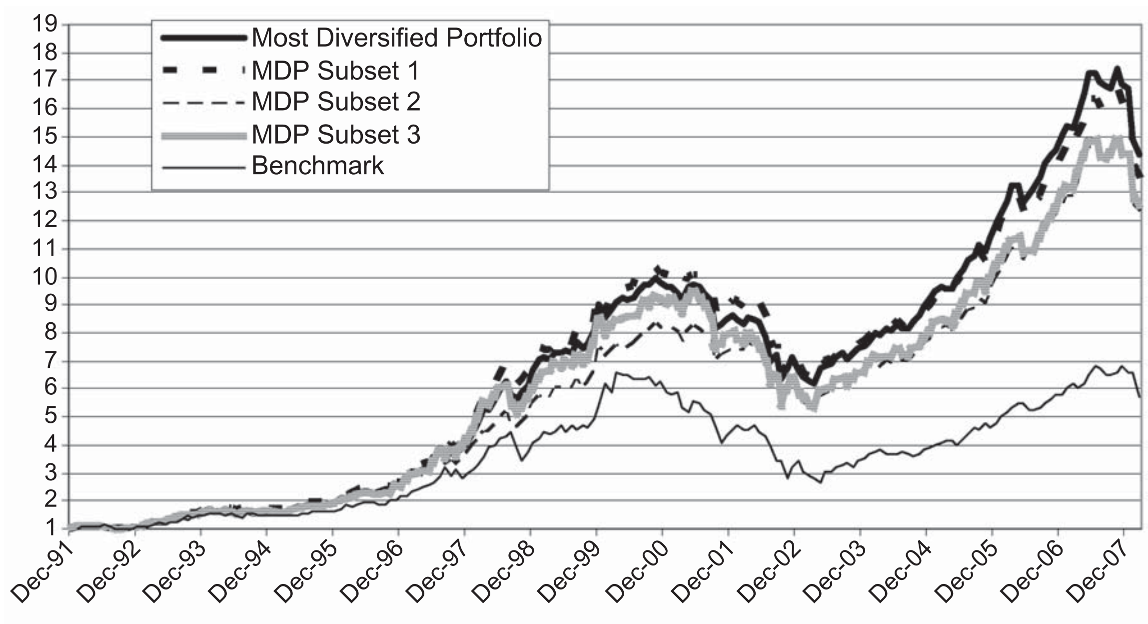
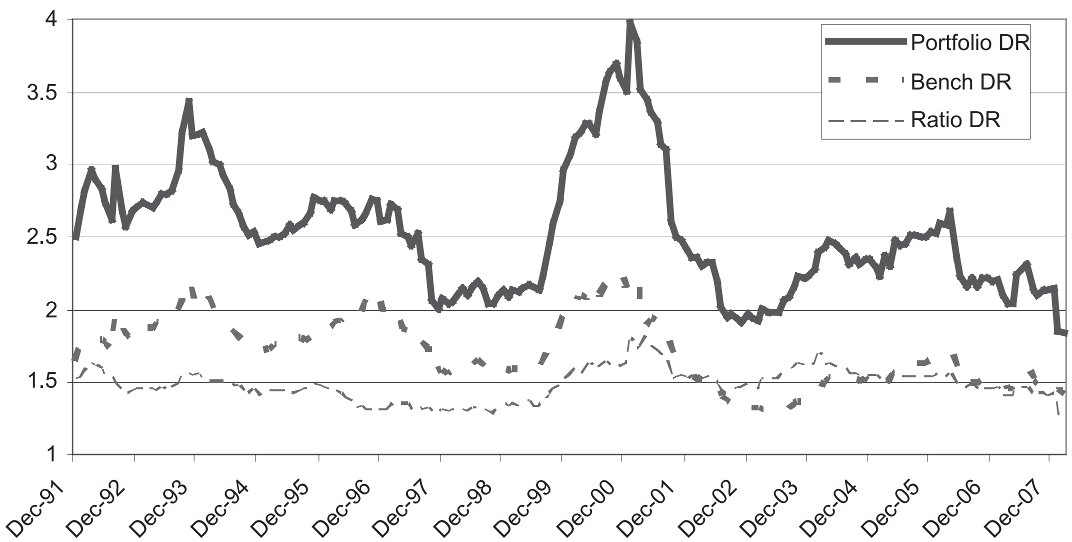


EXHIBIT 7

Comparison of Eurozone Most-Diversified Portfolio and Benchmark Diversification Ratios, 1992–2008



These assumptions alone, of course, are not enough for the Most-Diversified Portfolio to be an equilibrium model.

CONCLUSION

In this article, we provide a mathematical definition of diversification and describe several implications of diversification as a goal. Most-Diversified Portfolios have higher Sharpe ratios than the market cap–weighted indices and have had both lower volatilities and higher returns in the long run, which can be interpreted as capturing a bigger part of the risk premium.

Empirical results tend to confirm the value of a theoretical framework for diversification. It is difficult to determine if a portfolio was ex ante on the efficient frontier, but evidence tends to indicate that the Most-Diversified Portfolio is more efficient ex post than the market cap–weighted benchmark, minimum-variance portfolio, and equal-weight portfolio.

Because the hypotheses in our analysis are not specific to the equity market, the Most-Diversified-Portfolio methodology can be adapted to other asset classes. And the diversification ratio can be viewed as a new measure of risk that, when combined with the performance of the Most-Diversified Portfolio, has explanatory power for the performance of any portfolio within the same universe of securities.

The goal of the Most-Diversified Portfolio is not to be an equilibrium model. It can, however, potentially be transformed into an equilibrium model either by adding additional assumptions or by adding fundamental valuation criteria, such as earnings, sales, and so forth. Such additions would allow the model to accommodate different mispricings.

We have defined a portfolio construction methodology that can be considered an alternative to other non-market-cap benchmarks (see, for example, Fernholz and Shay [1982] and Arnott, Hsu, and Moore [2005]), and, as such, is a new investment style that favors diversification and avoids bets based on return prediction or confidence in the implicit bets of market cap–weighted benchmarks.

ENDNOTE

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