International Journal of Theoretical and Applied Finance
Vol. 22, No. 7 (2019) 1950034 (52 pages)

DOI: 10.1142/S0219024919500341

PORTFOLIO RHO-PRESENTATIVITY

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Received 14 December 2017
Revised 1 March 2019
Accepted 17 June 2019
Published 7 November 2019

Given an investment universe, we consider the vector $\rho(w)$ of correlations of all assets to a portfolio with weights w. This vector offers a representation equivalent to w and leads to the notion of ρ-presentative portfolio, that has a positive correlation, or exposure, to all assets. This class encompasses well-known portfolios, and complements the notion of representative portfolio, that has positive amounts invested in all assets (e.g. the market-cap index). We then introduce the concept of maximally ρ-presentative portfolios, that maximize under no particular constraint an aggregate exposure $f(\rho(w))$ to all assets, as measured by some symmetric, increasing and concave real-valued function f. A basic characterization is established and it is shown that these portfolios are long-only, diversified and form a finite union of polytopes that satisfies a local regularity condition with respect to changes of the covariance matrix of the assets. Despite its small size, this set encompasses many well-known and possibly constrained long-only portfolios, bringing them together in a common framework. This also allowed us characterizing explicitly the impact of maximum weight constraints on the minimum variance portfolio. Finally, several theoretical and numerical applications illustrate our results.

Keywords: Portfolio construction; correlation optimization; constraints; representative portfolios; diversification; maximally rho-presentative portfolios; optimized portfolio stability; long-only eigenvalues.

1. Introduction

For more than a decade, new quantitative investment processes delivering an exposure to the overall market have attracted significant interest in the field of asset management. We briefly present some of these long-only strategies whose main input is the covariance matrix of the assets, and show how they could be rediscovered in the context of the present paper.

A simple portfolio delivering such an exposure that is different from the market capitalization-weighted index is the equally-weighted portfolio (hereinafter EW).

This particular choice is not new, with DeMiguel *et al.* (2009) claiming that it dates back to 400 AD. Using volatility-adjusted weights as an *alternative representation* for a portfolio naturally leads to the concept of equal-volatility-weighted portfolio (denoted by EVW). The relative contribution of each asset to the risk of a portfolio gives another way of representing it, and leads to the concept of portfolio that equalizes these risk contributions or ERC [see Maillard *et al.* (2010) and Spinu (2013)]. Following a very different path, in Fundamental Indexation (Arnott *et al.* 2005), the authors proposed equity portfolios with weights proportional to key accounting measures such as sales, book value and earnings. Such a portfolio is *representative* of a universe in the sense that it invests in each company in proportion to its "economic footprint" rather than its capitalization.

As we have seen, approaches such as the EW, EVW, ERC and Fundamental Indexation emerge using alternative representations for portfolio weights, without explicitly using expected returns information. As we shall see in this paper, (possibly constrained) optimized portfolios such as the minimum variance [or MV; see Markowitz (1952)] and the most diversified portfolio [henceforth denoted by MDP; see Choueifaty & Coignard (2008) and Choueifaty *et al.* (2013)] can also be obtained through the representation of a portfolio by its correlations — or exposures — to all assets.

Finally, when reaching the implementation phase, these long-only investment processes may be modified in a number of ways. An important consideration for portfolios that optimize a given objective function is for example the addition of maximum weight constraints. These are imposed by some regulators and implemented by practitioners, and it is important to understand their impact on the initial objective. In Jagannathan & Ma (2003), it is shown that imposing such constraints for the MV problem is equivalent to minimizing an unconstrained variance objective using a modified covariance matrix. However, a limitation of the method is that the modified matrix depends on Lagrange multipliers that are either known after the MV optimization or determined through a numerically demanding optimization (a constrained max likelihood on matrices).

1.1. *Contributions of this paper*

A new portfolio representation using correlations. The usual representation of a portfolio, that consists in reporting its weights in each asset of the investment universe, may not directly indicate to which degree the portfolio is exposed to a particular asset. For instance, not holding any financial stock does not necessarily mean no exposure to the financial sector. This observation prompts a new representation of portfolios: given a long-short portfolio w, we consider in Sec. 2 its *correlation spectrum* $\rho(w)$ that stores its correlations to the assets of the universe and prove that it carries all the information needed to recover w up to its leverage.

Notions of representativity and ρ-presentativity. The capitalization-weighted portfolio is usually viewed as being representative of the assets of its universe. Such *representative portfolios* have positive amounts invested across all assets, leading us to introduce in Sec. 3 the notion of *ρ-presentative portfolio*, that admits a positive correlation, or exposure, to all assets. Note that this definition is not limited to long-only portfolios.

Optimized long-only portfolios such as the MDP and the MV are ρ-presentative without necessarily holding all the assets. In contrast, the market capitalization portfolio, the EW or the EVW, invested across all assets, are not necessarily ρ-presentative. Both categories intersect as, for instance, the ERC resides in both.

Furthermore, portfolios that are ρ-presentative satisfy a fundamental property that is not true in general: the (not necessarily unique) least correlated long-only portfolio to a ρ-presentative portfolio is an asset. Using this result, we prove that a long-only portfolio is always positively correlated to at least one asset and give a uniform lower bound for this correlation.

Maximally $\boldsymbol{\rho}$-presentative portfolios. To complement the notion of ρ-presentative portfolio, we introduce in Sec. 4 the central concept of *maximally ρ-presentative portfolio*. By definition, such a portfolio maximizes an aggregate exposure $f(\rho(w)) \in \mathbb{R}$ to all assets as measured by some increasing, symmetric and concave function f. We show that maximally ρ-presentative portfolios are long-only. To establish this result, the key is to prove that for any portfolio that is not long-only there always exists a long-only portfolio that is more correlated to all assets. In addition, we characterize explicitly the set of maximally ρ-presentative portfolios: these are essentially the long-only portfolios whose exposures form a nonincreasing function of their volatility-adjusted weights. This property implies in turn that these portfolios are diversified.

Furthermore, we show that these portfolios form a finite union of polytopes and are quite rare essentially because any permutation of a maximally ρ-presentative portfolio that is different from it is not maximally ρ-presentative.

Despite its small size, this new class encompasses many well-known portfolios. For instance the EVW is, amongst all long-short portfolios, the portfolio that maximizes its average correlation to all the assets. We also prove that the MDP is the portfolio that maximizes its minimal correlation to all the assets amongst all long-short portfolios. We refine this result by showing that the MDP maximizes its minimal correlation to all long-only factors, defined as factors that are replicable by possibly leveraged long-only portfolios of assets belonging to the universe. Similar results are established for the ERC, MV and EW portfolios.

On the impact of maximum weight constraints. We prove in Sec. 5.2 that a *constrained* MV or MDP problem with maximum weight $\frac{1}{k}$ is essentially equivalent to an *unconstrained* maximization of an average of the k smallest entries of $\rho(w)$. In addition to proving, for instance, that the constrained MDP is maximally ρ-presentative, this result characterizes the impact on the objective of these constraints and is therefore related to Jagannathan & Ma (2003). In our case, the objective is explicit and does not involve *a priori* unknown Lagrange multipliers.

A framework for constructing alternative strategies. We prove in Sec. 5.3 that well-known investment strategies — possibly constrained — maximize an unconstrained objective that is a function of the spectrum $\rho(w)$. As a result, we obtain a unifying framework for constructing alternative investment strategies, that maximize their overall exposure to all assets.

Stability of the set of maximally ρ-presentative portfolios. Relying on the particular structure of the set of maximally ρ-presentative portfolios, we prove in Sec. 6 that it is stable whenever the input varies. More precisely, we show that the distance between two covariance matrices controls locally the distance between the corresponding sets of maximally ρ-presentative portfolios. Doing so, we introduce the concept of *long-only eigenvalue* which is relevant to analyze the stability of long-only optimized portfolios.

Applications. In Sec. 7.1, we give a theoretical application of our results on constrained portfolios by extending the "Core Properties" of Choueifaty *et al.* (2013) to the constrained case. In Sec. 7.3, we perform a numerical experiment where we consider more than 2000 US funds with unknown composition to pinpoint those that qualify for being maximally ρ-presentative. Doing so we also derive a formula to compute the realized diversification of a fund with unknown composition, using time series only.

1.2. *Assumptions and notations*

Assumption. In this paper, we assume that the covariance matrix of the assets Σ is a positive-definite and symmetric real matrix. This yields a clear presentation at the cost of a slight loss of generality. To see this, observe that, using the limiting case of the Cauchy–Schwarz inequality, this hypothesis is sufficient to prove the following:

Proposition 1.1. *Two portfolios are identical up to leverage if and only if they are perfectly correlated.*

If Σ is only positive semi-definite, the proposition does not hold. However, the statements where we prove that portfolios are identical could be reformulated by claiming that they are perfectly correlated. As a result, in several places one can weaken our assumption without weakening significantly the assertions (see Remark 5.9 for a detailed discussion). Note also that we do not assume in this paper that Σ has nonnegative entries. This would have shortened some of our proofs (for instance in Secs. 5.1 or 5.2) but would be less relevant to covariances observed in broad financial markets. Lastly, in this paper, we do not discuss how Σ is computed in practice and the data that are used to do so.

Notations. We consider a universe of $n \geq 2$ assets and let Σ, C and σ denote their covariance matrix, correlation matrix and volatilities vector. These matrices are related by $\Sigma = D(\sigma)CD(\sigma)$, where $D(\sigma)$ is the diagonal matrix with σ as a diagonal. We denote the Euclidean inner product in $\mathbb{R}^n$ and its associated norm by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. The nonnegative $\sigma_\Sigma(w) := \langle \Sigma w, w \rangle^{\frac{1}{2}}$ is the *volatility of a portfolio* with weights $w \in \mathbb{R}^n$ and $\|w\|_1 = \sum_{i=1}^n |w_i|$ is its *leverage*. Given w with $\sigma_\Sigma(w) > 0$, its *Diversification Ratio* and its *correlation to a portfolio* x with $\sigma_\Sigma(x) > 0$ are defined by

$$\mathrm{DR}_\Sigma(w) := \frac{\langle w, \sigma \rangle}{\sigma_\Sigma(w)} \quad \text{and} \quad \varrho_\Sigma(w, x) := \frac{\langle \Sigma w, x \rangle}{\sigma_\Sigma(w)\sigma_\Sigma(x)}. \tag{1.1}$$

The subscript indicates that the matrix Σ is used for the calculations, and will be omitted when clear from the context. Let us also introduce the set of *long-short unlevered* portfolios and its *long-only unlevered* version:

$$\Pi := \{w \in \mathbb{R}^n / \|w\|_1 = 1\} \quad \text{and} \quad \Pi^+ := \{w \in \Pi / \forall\, i \in \{1, \ldots, n\}, w_i \geq 0\}. \tag{1.2}$$

It is important to note that, *up to leverage, any nonzero long-short portfolio is represented within* Π.

To simplify our calculations, we shall denote the entrywise multiplication (respectively, division) between matrices by $\odot$ (respectively, $\oslash$). Whenever we write that a matrix $\Sigma \succ 0$ (respectively, $\Sigma \succeq 0$), we mean that it is positive-definite (respectively, positive semi-definite) whereas when two vectors $x, y \in \mathbb{R}^n$ are such that $x \succ y$ (respectively, $x \succeq y$), it means that $\forall i, x_i > y_i$ (respectively, $\forall i, x_i \geq y_i$). For any $x \in \mathbb{R}^n$, let $x_{(i)}$ denote the ith-order statistic of x, that is defined by the reordering $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. Alternatively, let $x^\uparrow$ (respectively, $x^\downarrow$) denote the vector that contains the elements of x sorted in nonincreasing (respectively, nondecreasing) order. Talking about orderings, a function $f : \mathbb{R}^n \to \mathbb{R}$ is *increasing* or *order preserving* if $x \succ y$ implies that $f(x) > f(y)$ for any $x, y \in \mathbb{R}^n$. Such a function, if continuous, is also *nondecreasing* since whenever $x \succeq y$, one has $f(x) \geq f(y)$ for any $x, y \in \mathbb{R}^n$.

We quickly review the portfolios we consider in this paper starting with the EW and the EVW defined by $w_{\mathrm{ew}} = \mathbf{1}/n$ and $w_{\mathrm{evw}} = \frac{\mathbf{1} \oslash \sigma}{\langle \mathbf{1}, \mathbf{1} \oslash \sigma \rangle}$. The MV w_{mv} minimizes σ_Σ over Π^+ and the ERC solves for $w_{\mathrm{erc}} \odot (\Sigma w_{\mathrm{erc}}) = n^{-1}\sigma^2(w_{\mathrm{erc}})\mathbf{1}$ in Π^+. The long-only MDP w^* maximizes DR_Σ over Π^+. Abusing notations, we call long-short "MDP" the portfolio $\bar{w} := \Sigma^{-1}\sigma / \|\Sigma^{-1}\sigma\|_1$ that maximizes DR_Σ over Π and we always refer to the long-only portfolio when using MDP alone. In addition, we consider the market capitalization-weighted portfolio denoted by MKT, and a long-short portfolio called PCA and denoted by w_{pca} defined as any eigenvector of Σ. We refer to the aforementioned literature for discussions on the existence, uniqueness and other properties of these portfolios.

2. A New Portfolio Representation Using Correlations

A key aspect of this paper is the use of an alternative portfolio representation that takes into account *the exposure of a portfolio to all assets* of the investment universe.

A candidate for such a representation is the concept of correlation spectrum that we present in this section.

2.1. *Definition and key property of the correlation spectrum*

Definition 2.1. The *correlation spectrum* of a portfolio with weights $w \in \mathbb{R}^n \backslash \{0\}$ is the vector $\rho_\Sigma(w) \in \mathbb{R}^n$ such that for any index $i \in \{1, \ldots, n\}$

$$\rho_\Sigma(w)_i := \varrho_\Sigma(w, e_i), \tag{2.1}$$

where $e_i \in \Pi^+$ is the single-asset portfolio invested in the asset i.

In other words, $\rho_\Sigma(w) = \sigma_\Sigma(w)^{-1}(\Sigma w) \oslash \sigma$. Note that specializing $\Sigma = C$, one has $\rho_C(w) = \sigma_C(w)^{-1} C w$. We may omit the subscript Σ and write $\rho(w)$ instead of $\rho_\Sigma(w)$ when it is perfectly clear that this *particular* covariance matrix is used in the calculation.

The correlation spectrum *alone* allows to compare the signed exposures of a given portfolio to each asset in the universe. Consider for example a portfolio that has a positive correlation to asset a that is twice that to asset b: a positive one standard deviation return of either asset can be expected to result in a positive portfolio return that is twice as large for asset a than for asset b. Note that another measure of exposure, namely the *marginal risk contribution* [see Roncalli (2013)], will be briefly considered in Sec. 5.2.2.

Finally, we show that, given a fixed leverage, it is equivalent to represent a long-short portfolio by its weights or by its correlation spectrum.

Proposition 2.2. *The mapping* $w \in \Pi \mapsto \rho(w) \in \mathcal{E} := \{z \in \mathbb{R}^n, \langle C^{-1} z, z\rangle = 1\}$ *is bijective.*

Proof. For $w \in \Pi$, $\langle C^{-1}\rho(w), \rho(w)\rangle = \sigma(w)^{-2}\langle \Sigma^{-1}\Sigma w, \Sigma w\rangle = 1$. Furthermore, given $z \in \mathcal{E}$, we define $\rho^{-1}(z) := \Sigma^{-1}(z \odot \sigma)/\|\Sigma^{-1}(z \odot \sigma)\|_1 \in \Pi$ and verify readily that $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho = I$. □

Example 2.3. To illustrate our definition, we pick the MSCI USA universe and plot in Fig. 1 the *independently* sorted vectors $\rho(w)^{\downarrow}$ associated to the EVW, MV, ERC, MDP, long-short MDP and MKT portfolios.

2.2. *Other properties of the correlation spectrum*

The following proposition contains a composition formula that gives the expression of the spectrum of the convex combination of two long-short portfolios as a function of their individual spectra.

Proposition 2.4. *Take two different* $w_0, w_1 \in \Pi$ *and* $\theta \in (0, 1)$ *with* $w_\theta := \theta w_1 + (1-\theta) w_0 \in \mathbb{R}^n \backslash \{0\}$. *Then,*

$$\rho(w_\theta) = d_\theta(\mu_\theta \rho(w_1) + (1 - \mu_\theta)\rho(w_0)), \tag{2.2}$$

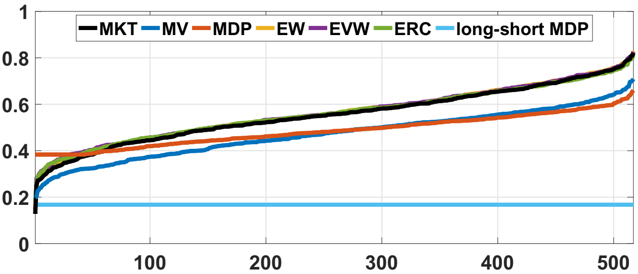

Fig. 1. Correlation spectra sorted *independently* for the MDP, long-short MDP, MV, ERC, EVW and MKT portfolios. The x-axis represents the rank of a constituent of the MSCI USA and the y-axis the correlation over January 2014–March 2017 of a portfolio to that stock. The flat region for the MDP spectrum was mentioned in the First Core Property of Choueifaty *et al.* (2013). Indeed, the MDP is more correlated to the stocks it does not hold than to those it holds and it has the same correlation to the latter ones (see Sec. 7.1 in this paper for a generalization).

with

$$d_\theta = \frac{\theta\sigma(w_1) + (1-\theta)\sigma(w_0)}{\sigma(w_\theta)} > 1 \quad \textit{and} \quad \mu_\theta = \frac{\theta\sigma(w_1)}{\theta\sigma(w_1) + (1-\theta)\sigma(w_0)} \geq 0.$$

Proof. As $w_\theta \neq 0$,

$$\rho(w_\theta) = \sigma(w_\theta)^{-1}\Sigma(\theta w_1 + (1-\theta)w_0) \oslash \sigma \tag{2.3}$$

$$= \sigma(w_\theta)^{-1}(\theta\sigma(w_1)\rho(w_1) + (1-\theta)\sigma(w_0)\rho(w_0)). \tag{2.4}$$

Since $w_1 \neq w_0$ and σ_Σ is strictly convex then $d_\theta > 1$. □

This result is reminiscent of the diversification axiom in Artzner *et al.* (1999) which states that for a coherent risk measure, the risk associated with a weighted combination of assets is no larger than the weighted combination of the individual risks of the assets. Indeed, the scaling d_θ that appears in the above formula measures exactly such an effect. A version of this proposition for an arbitrary number of portfolios is provided in Appendix A.1.

In the rest of the paper, we shall use the mapping $\phi : \Pi^+ \to \Pi^+$ defined by $\phi(w) := \frac{1}{\langle w,\sigma\rangle} w \odot \sigma$. It is a bijection and, with $x = \phi(w)$, its inverse is given by $w = \phi^{-1}(x) := \frac{1}{\langle x, \mathbf{1}\oslash\sigma\rangle} x \oslash \sigma$. As shown in the following proposition, the function ϕ is helpful as it allows treating assets as if they had identical volatilities.

Proposition 2.5. *The function ϕ is a bijection from $\Pi^+ \to \Pi^+$ and given two portfolios $w_1, w_2 \in \Pi^+$,*

$$\rho_\Sigma(w_1) = \rho_C(x_1), \tag{2.5}$$

$$\varrho_\Sigma(w_1, w_2) = \varrho_C(x_1, x_2), \tag{2.6}$$

$$\sigma_\Sigma(w_1) = \langle \mathbf{1}, x_1 \oslash \sigma\rangle^{-1}\sigma_C(x_1), \tag{2.7}$$

$$\mathrm{DR}_\Sigma(w_1) = \sigma_C(x_1)^{-1} \geq 1. \tag{2.8}$$

Proof. The function ϕ is well defined on Π^+ as $\langle w, \sigma \rangle > 0$ over this set, the same goes for ϕ^{-1}. Then it is easy to check that $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = I$, and as a result that $\langle w, \sigma \rangle \langle x, \mathbf{1} \oslash \sigma \rangle = 1$. Note that, in the definition of ϕ, we simply need that the considered portfolios are not orthogonal to σ which, here, is implied by $\Sigma \succ 0$. Recalling that $D(\sigma)$ is the diagonal matrix with σ on the diagonal, one has $\Sigma = D(\sigma) C D(\sigma)$. Furthermore $\forall w \in \Pi^+, x = \phi(w) = \langle w, \sigma \rangle^{-1} D(\sigma) w$, so $D(\sigma) w = \langle x, \mathbf{1} \oslash \sigma \rangle^{-1} x$. Now, $\forall w_1, w_2 \in \Pi^+$,

$$\sigma_\Sigma^2(w_1) = \langle \Sigma w_1, w_1 \rangle = \langle C D(\sigma) w_1, D(\sigma) w_1 \rangle = \langle x_1, \mathbf{1} \oslash \sigma \rangle^{-2} \langle C x_1, x_1 \rangle, \quad (2.9)$$

with $\langle x_1, \mathbf{1} \oslash \sigma \rangle > 0$. Furthermore, for any $w_1 \in \Pi^+$,

$$\mathrm{DR}_\Sigma(w_1) = \frac{\langle w_1, \sigma \rangle}{\sigma_\Sigma(w_1)} = \frac{\langle x_1, \mathbf{1} \oslash \sigma \rangle}{\langle x_1, \mathbf{1} \oslash \sigma \rangle} \frac{1}{\sigma_C(x_1)} \geq 1, \quad (2.10)$$

since $\sigma_C(x_1) \leq 1$ given that $x_1 \in \Pi^+$. Moreover,

$$\varrho_\Sigma(w_1, w_2) = \frac{\langle \Sigma w_1, w_2 \rangle}{\sigma_\Sigma(w_1) \sigma_\Sigma(w_2)} \quad (2.11)$$

$$= \frac{\langle x_1, \mathbf{1} \oslash \sigma \rangle \langle x_2, \mathbf{1} \oslash \sigma \rangle}{\sigma_C(x_1) \sigma_C(x_2)} \langle D(\sigma) w_1, C D(\sigma) w_2 \rangle \quad (2.12)$$

$$= \varrho_C(x_1, x_2). \quad (2.13)$$

Lastly, since $\phi(e_i) = e_i$, the last identity implies that $\rho_\Sigma(w_1) = \rho_C(x_1)$. □

3. Notions of Representativity and ρ-presentativity

The capitalization-weighted index is usually regarded as "representative" of its investment universe, and has by definition a positive weight on each asset. This consideration leads to the following:

Definition 3.1. A portfolio $w \in \mathbb{R}^n$ is *representative* if $w \succ 0$.

This definition has some limitations as the weight of an asset in a portfolio may not accurately measure the exposure of the portfolio to that asset. Therefore, to compare the exposure of a portfolio to several stocks we may use a measure that relies on correlations, prompting the following definition.

Definition 3.2. A portfolio $w \in \mathbb{R}^n \backslash \{0\}$ is *ρ-presentative* if $\rho(w) \succ 0$.

This definition leaves the way open to long-short portfolios as such a portfolio may be ρ-presentative. Let us now give few examples of ρ-presentative portfolios.

Proposition 3.3. *The long-only ERC, MV and MDP are ρ-presentative. The EW, EVW, the market capitalization-weighted index MKT and a PCA portfolio are not necessarily ρ-presentative. The long-short Max Sharpe portfolio $\Sigma^{-1}\mu$ is ρ-presentative if and only if the excess expected returns μ are positive.*

Proof. Let us prove that the MDP is ρ-presentative. To do so let us first establish that

$$\underset{w\in\Pi^+}{\operatorname{argmax}}\,\mathrm{DR}(w) = \phi^{-1}\left(\underset{x\in\Pi^+}{\operatorname{argmin}}\,\sigma_C(x)\right). \tag{3.1}$$

Both $w \mapsto \mathrm{DR}(w)$ and $w \mapsto \rho(w)$ are well defined on Π^+. The continuity over the compact Π^+ of $w \mapsto \mathrm{DR}(w)$ and $x \mapsto \sigma_C(x)$ shows that there exist elements in Π^+ maximizing the former and minimizing the latter. Our claim follows from Proposition 2.5 that implies $\mathrm{DR}(w) = \sigma_C(x)^{-1}$ hence maximizing DR amounts to minimizing σ_C. As $C \succ 0$, $x^* = \phi(w^*)$ is unique and the same holds for w^*. Applying the KKT theorem [cf. Boyd & Vandenberghe (2004) and Rockafellar (1970)] to $\min_{x\in\Pi^+}\sigma_C(x)$ shows that the solution x^* solves $Cx^* = \sigma_C^2(x^*)\mathbf{1} + \lambda$, with $\lambda \odot x^* = 0$ and $\lambda \succeq 0$, hence $\rho_C(x^*) = \sigma_C(x^*)\mathbf{1} + \lambda\sigma_C(x^*)^{-1}$. So $\min \rho_C(x^*) = \sigma_C(x^*)$ as $x^* \in \Pi^+$ has a positive entry associated to a zero entry of λ. Finally, by Proposition 2.5, $\rho_\Sigma(w^*) = \rho_C(x^*)$ and $\sigma_C(x^*) = \mathrm{DR}(w^*)^{-1}$. To sum up,

$$\min \rho_\Sigma(w^*) = \min \rho_C(x^*) = \sigma_C(x^*) = \mathrm{DR}(w^*)^{-1} > 0. \tag{3.2}$$

In particular, the MDP w^* satisfies $\rho(w^*) \succeq \mathrm{DR}(w^*)^{-1}\mathbf{1} \succ 0$.

Similarly, by the MV first-order condition, $\Sigma w_{\mathrm{mv}} \succeq \sigma^2(w_{\mathrm{mv}})\mathbf{1}$, hence, $\rho(w_{\mathrm{mv}}) \succeq \sigma(w_{\mathrm{mv}}) \oslash \sigma \succ 0$. The long-only ERC solves $w_{\mathrm{erc}} \odot (\Sigma w_{\mathrm{erc}}) = n^{-1}\sigma^2(w_{\mathrm{erc}})\mathbf{1}$, hence, $\rho(w_{\mathrm{erc}}) = n^{-1}\sigma(w_{\mathrm{erc}}) \oslash (w_{\mathrm{erc}} \odot \sigma) \succ 0$. Note that this portfolio exhibits a nice feature if ρ-presentativity is the goal: the lower the correlation to an asset, the higher its weight. Taking the EW or the EVW portfolio of a large collection of highly correlated assets to which is added another asset sufficiently negatively correlated to the others proves that these portfolios are not ρ-presentative. The same argument holds in theory for the MKT portfolio. Lastly, observe that $\forall i \in \{1,\ldots,n\}$, $\mathrm{sgn}((\rho(w_{\mathrm{pca}}))_i) = \mathrm{sgn}((w_{\mathrm{pca}})_i)$ so w_{pca} is not necessarily ρ-presentative. □

The classes of representative and ρ-presentative portfolios intersect as the ERC lies in both. However, these two classes are not included in one another: as there exist representative portfolios that are not ρ-presentative, there are ρ-presentative portfolios that are not necessarily representative. The MDP is such a portfolio (cf. for instance Fig. 1). Lastly, the Max Sharpe portfolio is an example of a portfolio that is not necessarily long-only but that may happen to be ρ-presentative.

Let us pursue with a fundamental property of ρ-presentative portfolios.

Lemma 3.4. *Given a ρ-presentative portfolio w, the* (*not necessarily unique*) *least correlated long-only portfolio to w is an asset. Actually, for any $w \in \Pi$ such that $\rho(w) \succeq 0$,*

$$\min_{\theta\in\Pi^+} \varrho(w,\theta) = \min \rho(w). \tag{3.3}$$

This is based on the identity

$$\forall (w,\theta) \in (\mathbb{R}^n\backslash\{0\}) \times \Pi^+, \quad \varrho(w,\theta) = \mathrm{DR}(\theta)\langle\phi(\theta), \rho(w)\rangle. \tag{3.4}$$

Proof. The last identity follows from the generalized version of Proposition 2.4 that is in Appendix A.1 but we can also give a short and direct proof as $\forall (w,\theta) \in (\mathbb{R}^n \backslash \{0\}) \times \Pi^+$,

$$\varrho(w,\theta) = \frac{\langle w, \Sigma\theta\rangle}{\sigma(w)\sigma(\theta)} = \frac{\langle \theta \odot \sigma, \rho(w)\rangle}{\sigma(\theta)} = \frac{\langle \theta, \sigma\rangle}{\sigma(\theta)} \langle \phi(\theta), \rho(w)\rangle = \mathrm{DR}(\theta) \langle \phi(\theta), \rho(w)\rangle\,. \tag{3.5}$$

To prove the identity (3.3), observe that the infimum over θ is always smaller than the right-hand side so we just need to focus on the reverse inequality. As $\phi(\theta) \in \Pi^+$, $\forall z \in \mathbb{R}^n$, $\langle \phi(\theta), z\rangle \geq \min(z)$, and so

$$\varrho(w,\theta) = \mathrm{DR}(\theta) \langle \phi(\theta), \rho(w)\rangle \geq \mathrm{DR}(\theta) \min \rho(w) \geq \min \rho(w), \tag{3.6}$$

where we used $\mathrm{DR}(\theta) \geq 1$ and our assumption that guarantees that $\min \rho(w) \geq 0$. We conclude the proof of the identity by taking the minimum with respect to $\theta \in \Pi^+$, which exists by continuity of $\theta \mapsto \varrho(w,\theta)$ on Π^+.

In case $\rho(w) \succ 0$, assume that the min over θ is attained by $\theta^* \in \Pi^+$, then combining (3.6) and the fact that $\varrho(w,\theta^*) = \min \rho(w)$, one has $\mathrm{DR}(\theta^*) = 1$. As σ_Σ is strictly convex, this is possible only if θ^* is an asset. □

The lemma implies that, whenever all entries of Σ are positive, the least correlated long-only portfolio to another long-only portfolio is an asset. However, in general, one cannot drop the assumption $\rho(w) \succeq 0$ as one can build a counter-example with a matrix that has negative entries and where we can verify that

$$\min_{\theta \in \Pi^+} \varrho(w,\theta) < \min \rho(w) \tag{3.7}$$

for some portfolios w that are therefore not ρ-presentative. See Fig. 6 for such a counter-example that can only occur in a non-Euclidean setting where geodesics are different from segments. Indeed, in a Euclidean setting, given a point in a convex body, the not necessarily unique point that is furthest from it within the body is necessarily an extreme point.

A consequence of Lemma 3.4 is derived from its combination with the identity (3.2).

Proposition 3.5. *A long-only portfolio is positively correlated to at least one asset, since*

$$\min_{w \in \Pi^+} \max \rho(w) \geq [\min \rho(w^*)]^2 = [\mathrm{DR}(w^*)]^{-2} > 0, \tag{3.8}$$

where we recall that w^* denotes the MDP.

Proof. Given $w \in \mathbb{R}^n \backslash \{0\}$ and considering ϕ as defined before Proposition 2.5, $\phi(w^*) \in \Pi^+$ which implies that

$$\max \rho(w) \geq \langle \phi(w^*), \rho(w)\rangle = \mathrm{DR}(w^*)^{-1} \varrho(w,w^*) = \min \rho(w^*) \varrho(w,w^*), \tag{3.9}$$

where we applied (3.4) in Lemma 3.4. Then we take on both ends the minimum [which exists by continuity of $\max(\rho(\cdot))$ and $\rho(\cdot, w^*)$ on the compact Π^+] and as

$\rho(w^*) \succ 0$, we can apply (3.3) in Lemma 3.4:

$$\min_{w\in\Pi^+} \max \rho(w) \geq \min \rho(w^*) \min_{w\in\Pi^+} \varrho(w, w^*) = [\min \rho(w^*)]^2 > 0. \qquad (3.10)$$

□

As a weak converse of this result, observe that a portfolio $w \in \mathbb{R}^n\backslash\{0\}$ that is ρ-presentative cannot be short-only since both $\langle \Sigma w, w\rangle > 0$ and $\Sigma w \succ 0$. This direction will be explored further in the next section.

4. Maximally ρ-presentative Portfolios

As we have seen in the previous section, it is possible to build portfolios that are ρ-presentative, i.e. that have a positive exposure to all assets. In this section, we introduce a complementary notion by considering portfolios that maximize their overall exposure to all assets.

4.1. *Definition and equivalent characterization*

Definition 4.1. A portfolio $w_f \in \mathbb{R}^n\backslash\{0\}$ is *maximally ρ-presentative* if there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ that is increasing, symmetric and concave such that

$$w_f \in \operatorname*{argmax}_{\mathbb{R}^n\backslash\{0\}} f \circ \rho. \qquad (4.1)$$

Let $\mathcal{R}$ denote the set of all unlevered maximally ρ-presentative portfolios.

A maximally ρ-presentative portfolio maximizes its exposures to all assets through an aggregate view offered by f which measures how ρ-presentative a portfolio is as a whole, given its exposures. Specifically:

(i) f is increasing to advantage a portfolio that is more ρ-presentative than another. In other words, if $\rho(w) \succ \rho(y)$ then $f \circ \rho(w) > f \circ \rho(y)$. This assumption excludes for instance $f = \|\cdot\|_2$.

(ii) f is concave which is consistent with the property of $\rho(w_\theta)$ in Proposition 2.4. Furthermore, we shall see that this assumption is key to prove that for fixed f, there is a unique maximally ρ-presentative portfolio. As a counter-example, for $f(x) = \sum_{i=1}^n x_i^3$ and $\Sigma = I$, optima of (4.1) are the single-asset portfolios.

(iii) f is symmetric, i.e. invariant under a permutation of coordinates, as there is *a priori* no reason for it to change if we permute the exposures. This excludes $f = \langle \cdot, \theta\rangle$ with $\theta \in \Pi^+\backslash\{n^{-1}\mathbf{1}\}$.

The examples we just gave will be further discussed in Sec. 5.3 and we shall see in the sequel how the concepts of ρ-presentative and maximally ρ-presentative portfolios compare to each other.

Let us now establish the existence and uniqueness of a such a portfolio for a given f.

Proposition 4.2. *For a concave increasing* f, *the maximum in* (4.1) *is reached by a unique unlevered portfolio.*

Proof. Since $\rho_\Sigma : \Pi \to \mathcal{E} = \{z, \|z\|_{C^{-1}} = 1\}$ is a bijection, one has $\sup_\Pi f \circ \rho = \sup_\mathcal{E} f$ with the left problem having a unique maximum if and only if the same is true for the right one, so we may focus on the latter one.

Existence: As in finite dimension, any concave function is continuous in the interior of its domain [see Theorem 10.1 in the monograph of Rockafellar (1970)], f attains its supremum m^* on the compact ball $\mathcal{E}$.

Uniqueness: Assuming the contrary, there are $z_1 \neq z_2$ such that $\|z_1\|_{C^{-1}} = \|z_2\|_{C^{-1}} = 1$ and $f(z_1) = f(z_2) = m^*$. Then considering a strict convex combination z_θ of z_1 and z_2, $\|z_\theta\|_{C^{-1}} < 1$ by strict convexity of the norm whereas by concavity of f, $f(z_\theta) \geq m^*$. Since $\lambda \mapsto \|z_\theta + \lambda \mathbf{1}\|_{C^{-1}}$ is continuous on $[0, +\infty)$ and tends to $+\infty$ when $\lambda \to +\infty$, then by the intermediate value theorem $\exists \lambda^* \in (0, +\infty)/\|z_\theta + \lambda^* \mathbf{1}\|_{C^{-1}} = 1$. On the other hand, since f is increasing, $f(z_\theta + \lambda^* \mathbf{1}) > f(z_\theta) \geq m^*$, hence a contradiction with the definition of m^*. □

The following result and ensuing theorem show that long-only portfolios have a special role amongst long-short portfolios seeking to maximize their exposure to all assets.

Lemma 4.3. *For any* $y \in \Pi \backslash \Pi^+$, *there exists* $w \in \Pi^+$ *such that* $\rho(w) \succ \rho(y)$. *However, this cannot hold for long-only portfolios. Indeed, if* $w \in \Pi, y \in \Pi^+$ *and* $\rho(w) \succeq \rho(y)$ *then* $w = y$.

Proof. To prove the first statement, consider the convex problem $\min_\mathcal{C} \sigma_\Sigma^2$ with $\mathcal{C} = \{z \in \mathbb{R}^n / z \succeq 0, \Sigma z \succeq \Sigma y\}$. It is feasible as we may always consider a rescaled enough long-only ρ-presentative portfolio and admits a unique solution v since its objective is strictly convex and the constraints are linear. Without loss of generality, one can assume that $v \neq 0$ as otherwise $\Sigma y \preceq 0$ and any long-only unlevered ρ-presentative w satisfies our first statement.

If we take $\lambda, \mu \succeq 0$ to be the Lagrange multipliers associated to the constraints $v \succeq 0$ and $\Sigma(v - y) \succeq 0$, the solution v solves the following KKT conditions: $\Sigma v = \lambda + \Sigma \mu$, $\lambda \odot v = 0$ and $\mu \odot (\Sigma(v - y)) = 0$. The first two conditions imply that $\sigma^2(v) = \langle \Sigma v, \mu \rangle$ and $\langle \Sigma v, \mu \rangle = \langle \lambda, \mu \rangle + \sigma^2(\mu) \geq \sigma^2(\mu)$ while the last one implies $\langle \Sigma v, \mu \rangle = \langle \Sigma y, \mu \rangle$. Then, as a result of the Cauchy–Schwarz inequality,

$$\sigma^2(v) = \langle \Sigma v, \mu \rangle = \langle \Sigma y, \mu \rangle \leq \sigma(y)\sigma(\mu) \leq \sigma(y)\langle \Sigma v, \mu \rangle^{\frac{1}{2}} = \sigma(y)\sigma(v). \qquad (4.2)$$

Since $v \neq 0$, we have $\sigma(v) > 0$ and therefore $\sigma(v) \leq \sigma(y)$.

Let us prove that $\sigma(v) = \sigma(y)$ cannot occur. If the identity holds then all inequalities in (4.2) are equalities hence $\langle \lambda, \mu \rangle = 0$ and $\langle \Sigma y, \mu \rangle = \sigma(y)\sigma(\mu)$. Then by the limiting case of the Cauchy–Schwarz inequality, there exists $\gamma \in \mathbb{R}$ such that $y = \gamma\mu$. Combining this observation and the fact that $\langle \lambda, \mu \rangle = 0$ with the first KKT condition yields $\langle \Sigma v, y \rangle = \langle \lambda, y \rangle + \langle \Sigma\mu, y \rangle = \gamma\langle \lambda, \mu \rangle + \langle \Sigma y, \mu \rangle = \langle \Sigma y, \mu \rangle$. Put together with (4.2) this implies that $\langle \Sigma v, y \rangle = \sigma(v)\sigma(y)$, i.e. $\varrho(v, y) = 1$, and thus $y = v \in \Pi^+$. This contradicts our assumption on y.

Therefore one necessarily has $\sigma(v) < \sigma(y)$. Consider a long-only ρ-presentative u that was rescaled enough so that $u \in \mathcal{C}$, $\Sigma u \succ \Sigma v \succeq \Sigma y$ and $\sigma(u) > \sigma(y) > \sigma(v)$. By continuity of σ_Σ on $\mathcal{C}$, there exists by the intermediate value theorem a strict convex combination $w = \alpha u + (1-\alpha)v \in \mathcal{C}$ with $\alpha \in (0, 1)$ such that $\sigma(w) = \sigma(y)$. Then, as $\Sigma w = \alpha\Sigma u + (1-\alpha)\Sigma v \succ \Sigma y$, one has $\frac{1}{\sigma(w)}\Sigma w = \frac{1}{\sigma(y)}\Sigma w \succ \frac{1}{\sigma(y)}\Sigma y$ hence $\rho(w) \succ \rho(y)$ (the proof is constructive as α is the root of a quadratic equation that one can readily compute).

The second statement of the lemma is obtained by taking the inner product with y and dividing by $\sigma(y)$ in the inequality $\Sigma w/\sigma(w) \succeq \Sigma y/\sigma(y)$. This implies that $\varrho(w, y) \geq 1$ hence $w = y$ by Proposition 1.1. □

Before getting further we recall that for any $v \in \mathbb{R}^n$, $v^\uparrow$ (respectively, $v^\downarrow$) denotes the vector that contains the elements of v sorted in nonincreasing (respectively, nondecreasing) order and that the bijective $\phi : \Pi^+ \to \Pi^+$ is defined by $\phi(w) := \frac{1}{\langle w,\sigma \rangle} w \odot \sigma$. Moreover we define $(\Pi^+)^\uparrow := \{w \in \Pi^+ / 0 \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq 1\}$. Equipped with these notations and Lemma 4.3 we are ready to prove the main result of this section.

Theorem 4.4.

(i) *Maximally ρ-presentative portfolios are exactly the portfolios $w \in \Pi^+$ that satisfy*

$$\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle = \langle \phi(w), \rho(w) \rangle. \tag{4.3}$$

In particular, as $\langle \phi(w), \rho(w) \rangle = \mathrm{DR}(w)^{-1}$,

$$\mathcal{R} = \underset{w \in \Pi^+}{\operatorname{argmax}}(\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle \mathrm{DR}(w)). \tag{4.4}$$

Said otherwise, maximally ρ-presentative portfolios are those long-only portfolios w such that there exists a permutation p_w of the assets that sorts their volatility-adjusted weights $\phi(w)$ in nondecreasing order and their exposures $\rho(w)$ in nonincreasing order.

(ii) *Given $\theta \in \Pi^+$ and $f_\theta : z \in \mathbb{R}^n \mapsto \langle \phi(\theta)^\uparrow, z^\downarrow \rangle$, the mapping*

$$\mathcal{P}_\mathcal{R} : \theta \in \Pi^+ \mapsto \underset{\Pi}{\operatorname{argmax}}\, f_\theta \circ \rho \tag{4.5}$$

is well defined and we have

$$\forall \theta \in \Pi^+, \quad \mathcal{P}_\mathcal{R}(\theta) \in \mathcal{R} \quad \textit{and} \quad \mathcal{P}_\mathcal{R}(\theta) = \theta \quad \textit{if and only if}\ \theta \in \mathcal{R}. \tag{4.6}$$

In addition, $\mathcal{P}_\mathcal{R} \circ \phi^{-1} : (\Pi^+)^\uparrow \to \mathcal{R}$ is surjective onto the set of maximally ρ-presentative portfolios.

(iii) *Representing maximally ρ-presentative portfolios w by their volatility-adjusted weights $\phi(w) \in \Pi^+$, their $(n-1)$-dimensional Lebesgue measure λ_{n-1} is such that*

$$\lambda_{n-1}(\phi(\mathcal{R})) \leq \frac{\lambda_{n-1}(\Pi^+)}{n!}. \tag{4.7}$$

(iv) *The set $\mathcal{R}$ is a finite union of polytopes.*

Proof. Let us start by proving that maximally ρ-presentative portfolios are long-only. Consider $y_f \in \operatorname{argmax}_\Pi f \circ \rho$ for some f that is continuous and increasing. Then $y_f \in \Pi^+$ as otherwise, by Lemma 4.3, $\exists w_f \in \Pi^+$ such that $\rho(w_f) \succ \rho(y_f)$, hence $f \circ \rho(w_f) > f \circ \rho(y_f)$ and y_f is not optimal.

Now, remark that by the identity (3.4) in Lemma 3.4,

$$\forall \theta \in \Pi^+, \quad \max_{w \in \Pi} \langle \phi(\theta), \rho(w) \rangle = \langle \phi(\theta), \rho(\theta) \rangle = \mathrm{DR}^{-1}(\theta) > 0, \tag{4.8}$$

which means that $\phi(\theta)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(\theta)$.

We pursue by proving that any maximally ρ-presentative w_f satisfies (4.3) and we refer to Fig. 2 for the geometric intuition behind the argument. Considering $\mathfrak{S}_n$, the group of permutations of $\{1, \cdots, n\}$, let us first note that for any $w \in \Pi^+$,

$$\langle \phi(w)^\uparrow, \rho(w)^\downarrow \rangle = \min_{p \in \mathfrak{S}_n} \langle \phi(w), p \circ \rho(w) \rangle. \tag{4.9}$$

Assuming that (4.3) does not hold, there exists $p \in \mathfrak{S}_n$ with $\langle \phi(w_f), p \circ \rho(w_f) - \rho(w_f) \rangle < 0$. Since $\phi(w_f)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(w_f)$, there exists a strict convex combination z_μ of $p \circ \rho(w_f)$ and $\rho(w_f)$ that lies in the interior of the domain enclosed by $\mathcal{E}$ since

$$\|z_\mu\|^2_{C^{-1}} = 1 + 2\mu \langle \rho(w_f), p \circ \rho(w_f) - \rho(w_f) \rangle_{C^{-1}} + \mu^2 \|p \circ \rho(w_f) - \rho(w_f)\|^2_{C^{-1}} \tag{4.10}$$

$$= 1 + 2\mu \mathrm{DR}(w_f) \langle \phi(w_f), p \circ \rho(w_f) - \rho(w_f) \rangle + \mu^2 \|p \circ \rho(w_f) - \rho(w_f)\|^2_{C^{-1}}, \tag{4.11}$$

which is smaller than 1 for $\mu > 0$ sufficiently small. We may then conclude as in the proof of the uniqueness in Proposition 4.2. Indeed, since f is concave and symmetric $f(z_\mu) \geq f \circ \rho(w_f)$. Then by the intermediate value theorem $\exists \lambda^* \in (0, +\infty)$ such that $z_\mu + \lambda^* \mathbf{1} \in \mathcal{E}$. Thus $\exists y \in \Pi$ such that $\rho(y) = z_\mu + \lambda^* \mathbf{1}$. Since f is increasing, $f \circ \rho(y) > f(z_\mu) \geq f \circ \rho(w_f)$ contradicting the optimality of w_f. Thus w_f satisfies (4.3).

Conversely, given $\theta \in \Pi^+$, we consider the function $f_\theta : z \mapsto \min_{p \in \mathfrak{S}_n} \langle \phi(\theta), p(z) \rangle$. This mapping as well as (4.5) are well defined as on the one hand we take the minimum over a finite number of permutations and on the other hand the objective in (4.5) is continuous over the compact Π. Moreover f_θ is increasing, symmetric and concave on $\mathbb{R}^n$ and $f_\theta(z) = \langle \phi(\theta)^\uparrow, z^\downarrow \rangle$. So if we take $\theta \in \Pi^+$ that

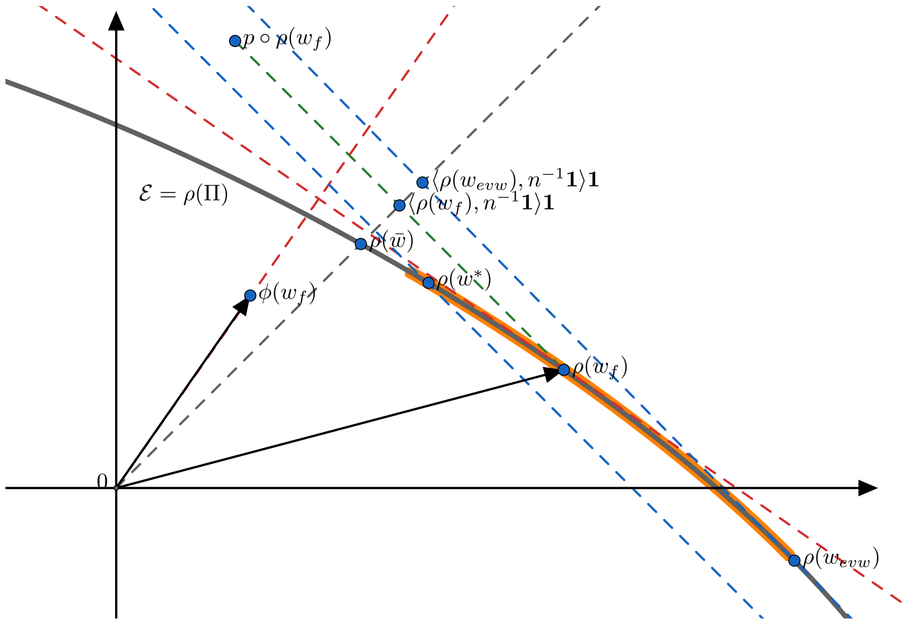

Fig. 2. A schematic view of the geometry behind the proofs and results of Propositions 4.2 and 4.5 and Theorem 4.4. The dashed lines in red are orthogonal as $\phi(w_f)$ is an outer normal to the ellipsoid $\mathcal{E}$ at $\rho(w_f)$. The green dashed segment is the set of convex combinations of the permutations $p \circ \rho(w_f)$. The set $\rho(\mathcal{R})$ of the spectra of maximally ρ-presentative portfolios *resides in* the orange region that is the intersection between the surface $\mathcal{E}$ and the strip of spectra whose averages are delimited by (4.31) and (5.1).

satisfies (4.3), then

$$\langle \phi(\theta), \rho(\theta)\rangle = f_\theta \circ \rho(\theta) \leq \max_{w\in\Pi} f_\theta \circ \rho(w) = \max_{w\in\Pi} \min_{p\in\mathfrak{S}_n} \langle \phi(\theta), p \circ \rho(w)\rangle \tag{4.12}$$

$$\leq \max_{w\in\Pi} \langle \phi(\theta), \rho(w)\rangle = \langle \phi(\theta), \rho(\theta)\rangle, \tag{4.13}$$

where in the two last steps we took $p = Id$ and used (4.8). Thus θ maximizes $f_\theta \circ \rho$ and so $\theta \in \mathcal{R}$.

By the previous analysis, for any portfolio $w \in \mathcal{R}$, $\exists x := \phi(w)^{\uparrow} \in (\Pi^+)^{\uparrow}$ such that $w = \operatorname{argmax}_{y\in\Pi}\langle x, \rho(y)^{\downarrow}\rangle$, hence the surjectivity.

Given $p \in \mathfrak{S}_n$, let $\Delta_p := \{w \in \Pi^+ / p\circ\phi(w) = \phi(w)^{\uparrow}, p\circ\rho(w) = \rho(w)^{\downarrow}\}$ be the set of portfolios whose volatility-adjusted weights and spectra are ordered in opposite directions by the same permutation p. Since $\Delta_p = \Pi^+ \cap \{w \in \mathbb{R}^n / p(w \odot \sigma) = (w\odot\sigma)^{\uparrow}, p(\Sigma w \oslash \sigma) = (\Sigma w \oslash \sigma)^{\downarrow}\}$, it is the intersection of the polytope Π^+ and of two sets of the form $T(\{w \in \mathbb{R}^n / p(w) = w^{\uparrow}\})$ and $S(\{w \in \mathbb{R}^n / p(w) = w^{\downarrow}\})$ with S and T being two linear and invertible mappings. The latter sets are therefore polytopes as they are images of two simplices by two linear mappings. Therefore Δ_p is also a polytope. Then observe that if not empty $\mathcal{R} \cap \{w \in \Pi^+ / p(\phi(w)) = \phi(w)^\uparrow\} = \Delta_p$ and $\mathcal{R}$ is a finite union of polytopes since $\mathcal{R} = \bigcup_{p \in \mathfrak{S}_n} \Delta_p$.

In the sequel, through the use of ϕ, we can assume that $\sigma = \mathbf{1}$. First, observe that for $w \in \mathcal{R}$ and any permutation p such that $p(w) \neq w$, $p(w) \notin \mathcal{R}$. Indeed, reasoning as in (4.12), we have that

$$\langle w, \rho(w)\rangle \le \max_{w' \in \Pi} \min_{q \in \mathfrak{S}_n} \langle w, q \circ \rho(w')\rangle \le \max_{w' \in \Pi} \langle w, p \circ \rho(w')\rangle \tag{4.14}$$

$$= \max_{w' \in \Pi} \langle p(w), \rho(w')\rangle = \langle p(w), \rho(p(w))\rangle. \tag{4.15}$$

So, if both $w, p(w) \in \mathcal{R}$, one has $\langle w^\uparrow, \rho(w)^\downarrow\rangle \le \langle p(w)^\uparrow, \rho(p(w))^\downarrow\rangle$ hence $\max_{\Pi^+} f_w \circ \rho = f_w \circ \rho(w) \le f_w \circ \rho(p(w))$. As by Proposition 4.2, f_w admits a unique maximum, $p(w) = w$. Note that this also tells us that if $w \in \mathcal{R}$, then for any permutation p such that $p(w) \neq w$, one has $\sigma_\Sigma(w) < \sigma_\Sigma(p(w))$.

We are now ready to prove that the measure of $\mathcal{R}$ is small as compared to that of Π^+. We remark that by (4.3) — that we have now proven — $\mathcal{R}$ is closed, hence λ_{n-1}-measurable for the $(n-1)$-dimensional Lebesgue measure λ_{n-1}. Now, let $\mathcal{N} \subset \Pi^+$ be the set of portfolios with each having at least two identical weights. Then its measure $\lambda_{n-1}(\mathcal{N}) = 0$. Thus, if $\mathcal{R}' = \mathcal{R} \setminus \mathcal{N}$, the set of maximally ρ-presentative portfolios that each have distinct coordinates, then

$$\lambda_{n-1}(\mathcal{R}) = \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}[\mathcal{R}' \cap \{w \in \Pi^+,\ p(w) = w^\uparrow\}] \tag{4.16}$$

$$= \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}[p(\mathcal{R}' \cap \{w \in \Pi^+, p(w) = w^\uparrow\})], \tag{4.17}$$

since permutations are isometries. Now as any permutation of $w \in \mathcal{R}$ that is distinct from it is not in $\mathcal{R}$, the measure of $\mathcal{R}$ is equal to the measure of the union of the *disjoint* sets $p(\mathcal{R}' \cap \{w \in \Pi^+,\ p(w) = w^\uparrow\})$ that all belong to $(\Pi^+)^\uparrow$ and is therefore smaller than $\lambda_{n-1}(\Pi^+)/n!$. $\square$

This theorem shows that the exposures of a maximally ρ-presentative portfolio form *essentially* a nonincreasing function of its volatility-adjusted weights. This theorem also shows that maximally ρ-presentative portfolios are rare amongst all long-only portfolios. For example, given n assets, if one drew uniformly the volatility-adjusted weights of N long-only portfolios, there are less than $\frac{N}{n!}$ chance to have drawn those of a maximally ρ-presentative portfolio.

4.2. *Maximally ρ-presentative portfolios are diversified*

The previous characterization allows to prove that maximally ρ-presentative portfolios are diversified in the sense that their Diversification Ratio is never less than that of an EVW portfolio. More precisely, we have the following proposition.

Proposition 4.5. *A maximally ρ-presentative portfolio w_f satisfies the bounds*

$$0 < \frac{\mathrm{DR}(w_{\mathrm{evw}})}{\varrho(w_f, w_{\mathrm{evw}})} \leq \mathrm{DR}(w_f) \leq \mathrm{DR}(w^*)\varrho(w_f, w^*), \tag{4.18}$$

where we recall that w^* denotes the MDP.
In terms of the objective f, we have the following bounds:

$$f(\mathrm{DR}(w^*)^{-1}\mathbf{1}) \leq f \circ \rho(w_f) \leq f(\mathrm{DR}(w_{\mathrm{evw}})^{-1}\mathbf{1}). \tag{4.19}$$

Proof of (4.18). By Proposition 2.5, the identity (3.4) in Lemma 3.4 and the characterization (4.3),

$$\mathrm{DR}^{-1}(w_f) = \min_{p \in \mathfrak{S}_n} \langle p \circ \phi(w_f), \rho_\Sigma(w_f)\rangle \tag{4.20}$$

$$= \min_{p \in \mathfrak{S}_n} \langle p(x_f), \rho_C(x_f)\rangle \tag{4.21}$$

$$\leq \langle \rho_C(x_f), n^{-1}\mathbf{1}\rangle \tag{4.22}$$

$$= \mathrm{DR}^{-1}(w_{\mathrm{evw}})\varrho(w_f, w_{\mathrm{evw}}). \tag{4.23}$$

Since by Theorem 4.4, $w_f \in \Pi^+$ then $\mathrm{DR}(w_f) \leq \mathrm{DR}(w^*)\varrho(w_f, w^*)$ by the Second Core Property in Choueifaty *et al.* (2013) (this latter result will be generalized in Proposition 7.2). This finishes the proof of (4.18). □

Proof of (4.19). Considering the shift operator $S(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$,

$$\forall w \in \mathbb{R}^n \backslash \{0\},\ f \circ \rho(w) = \frac{1}{n}\sum_{k=1}^{n} f \circ \rho(w) \tag{4.24}$$

$$= \frac{1}{n}\sum_{k=1}^{n} f(S^k\rho(w)) \tag{4.25}$$

$$\leq f\left(\frac{1}{n}\sum_{k=1}^{n} S^k\rho(w)\right) \tag{4.26}$$

$$= f(\langle \rho(w), n^{-1}\mathbf{1}\rangle\mathbf{1}) \tag{4.27}$$

$$= f(\langle \rho(w), w_{\mathrm{ew}}\rangle\mathbf{1}), \tag{4.28}$$

where we invoked the symmetry and the concavity. Then, invoking (3.4) in Lemma 3.4, for any $w \in \mathbb{R}^n \backslash \{0\}$,

$$f \circ \rho(w) \leq f(\langle \rho(w), w_{\mathrm{ew}}\rangle\mathbf{1}) = f(\langle \rho(w), \phi(w_{\mathrm{evw}})\rangle\mathbf{1}) = f(\mathrm{DR}(w_{\mathrm{evw}})^{-1}\varrho(w, w_{\mathrm{evw}})\mathbf{1}). \tag{4.29}$$

Therefore, since f is increasing, $\max_\Pi f \circ \rho \leq f(\mathrm{DR}(w_{\mathrm{evw}})^{-1}\mathbf{1})$ whereas on the other hand by (3.2) and (4.29),

$$f(\mathrm{DR}(w^*)^{-1}\mathbf{1}) = f(\min \rho(w^*)\mathbf{1}) \leq f \circ \rho(w^*) \leq \max_{\Pi} f \circ \rho \leq f(\langle \rho(w_f), n^{-1}\mathbf{1}\rangle \mathbf{1}). \tag{4.30}$$

Inequalities (4.29) and (4.30) are illustrated in Fig. 2. □

Portfolios that are ρ-presentative are exactly those that are positively correlated with all long-only portfolios. In general, a maximally ρ-presentative portfolio is not ρ-presentative. However, we shall see in the following definition and ensuing proposition how these two concepts come together.

Definition 4.6. A portfolio $w \in \mathbb{R}^n \backslash \{0\}$ is *weakly ρ-presentative* if its average exposure is positive, i.e. $\langle \rho(w), \mathbf{1}\rangle > 0$.

Proposition 4.7. *A maximally ρ-presentative portfolio w_f is weakly ρ-presentative and we have a bound for its average exposure that is uniform in f and that involves the MDP w^*:*

$$n^{-1}\langle \rho(w_f), \mathbf{1}\rangle \geq \mathrm{DR}^{-1}(w_f) \geq \min \rho(w^*) > 0. \tag{4.31}$$

Maximally ρ-presentative portfolios are positively correlated to a special long-only portfolio, namely the EVW. In particular,

$$\varrho(w_f, w_{\mathrm{evw}}) \geq \frac{\mathrm{DR}(w_{\mathrm{evw}})}{\mathrm{DR}(w^*)}. \tag{4.32}$$

Proof. Combining (3.2) and (4.18) we obtain (4.31). Also, as f is increasing, we get $n^{-1}\langle \rho(w_f), \mathbf{1}\rangle \geq \min \rho(w^*)$ directly from (4.30) without this time relying on Theorem 4.4. Inequality (4.18) yields directly (4.32). □

According to (4.18), portfolios reduced to assets — and whose DR equals one — are obviously long-only but never maximally ρ-presentative. In the following proof, we are going to consider another example of a long-only portfolio that is never maximally ρ-presentative, namely

$$w^\sharp \in \operatorname*{argmin}_{w \in \Pi^+} \varrho(w, w_{\mathrm{evw}}). \tag{4.33}$$

This portfolio may happen to be different from any asset as soon as the EVW is not ρ-presentative (see Lemma 3.4 and the remark below it). As indicated by the previous theorem, many long-only portfolios are not maximally ρ-presentative. In particular, considering the limiting case of inequality (4.32) with $w^* = w_{\mathrm{evw}}$, it appears that there is only one maximally ρ-presentative portfolio which is the EVW/MDP. This particular case occurs if and only if $\mathbf{1}$ is an eigenvector of the correlation matrix, as shown by the KKT conditions given by the MDP problem. Nonetheless, in general, $\mathcal{R}$ is not a singleton but it is not large either. In fact, in the following proposition, we show that there are large open regions of Π^+ with no maximally ρ-presentative portfolios.

Proposition 4.8. *The set of maximally ρ-presentative portfolios satisfies the following inclusions*:

$$\mathcal{R} \subset \mathcal{F} := \left\{ w \in \Pi^+, \varrho(w, w_{\text{evw}}) \geq \frac{\text{DR}(w_{\text{evw}})}{\text{DR}(w)} \right\} \tag{4.34}$$

$$\subset \tilde{\mathcal{F}} := \left\{ w \in \Pi^+, \varrho(w, w_{\text{evw}}) \geq \frac{\text{DR}(w_{\text{evw}})}{\text{DR}(w^*)} \right\} \subsetneq \Pi^+. \tag{4.35}$$

Moreover, both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are closed convex sets.

In particular, the tangent hyperplane to $\mathcal{F}$ at w_{evw} separates Π^+ into two sets such that one of them does not contain any maximally ρ-presentative portfolio.

Lastly, if the long-short MDP $\bar{w} \neq w^*$, the inclusion $\mathcal{R} \subset \tilde{\mathcal{F}}$ holds true even if we consider a strict inequality in the definition of $\tilde{\mathcal{F}}$.

Proof. Step 1. The inclusion $\mathcal{R} \subset \mathcal{F}$ follows from (4.18). Given $\bar{w} = \Sigma^{-1}\sigma$, we remark that for any $w \in \Pi$,

$$\text{DR}(\bar{w})\varrho(\bar{w}, w) = \frac{\langle \sigma, \Sigma^{-1}\sigma \rangle}{\sigma_\Sigma(\Sigma^{-1}\sigma)^2} \frac{\langle \Sigma\Sigma^{-1}\sigma, w \rangle}{\sigma_\Sigma(w)} = \frac{\langle \sigma, \Sigma^{-1}\sigma \rangle}{\langle \Sigma\Sigma^{-1}\sigma, \Sigma^{-1}\sigma \rangle} \frac{\langle \sigma, w \rangle}{\sigma_\Sigma(w)} = 1 \times \text{DR}(w). \tag{4.36}$$

Combining this identity with (4.18) yields $\forall w \in \mathcal{R}$, $\varrho(w_{\text{evw}}, \bar{w}) \leq \varrho(w, w_{\text{evw}})\varrho(w, \bar{w})$ which, letting $\lambda := \langle \Sigma w_{\text{evw}}, \bar{w} \rangle$, can be rewritten as $\lambda\sigma(w)^2 \leq \langle \Sigma w, w_{\text{evw}} \rangle \langle \Sigma w, \bar{w} \rangle = \frac{1}{4}(\langle w, \Sigma(\bar{w} + w_{\text{evw}}) \rangle^2 - \langle w, \Sigma(\bar{w} - w_{\text{evw}}) \rangle^2)$. Then considering the matrix $M := \lambda\Sigma + \frac{1}{4}(\Sigma(\bar{w} - w_{\text{evw}}))(\Sigma(\bar{w} - w_{\text{evw}}))'$ we may rewrite $\mathcal{F} = \{w \in \Pi^+, \ \|w\|_M \leq \langle w, \frac{1}{2}\Sigma(w_{\text{evw}} + \bar{w}) \rangle\}$ where we used the fact that $\forall w \in \mathcal{F}, \langle w, \Sigma(\bar{w} + w_{\text{evw}}) \rangle \geq 0$. Indeed, by definition of $\mathcal{F}$, $\langle w, \Sigma w_{\text{evw}} \rangle \geq 0$ and $\forall w \in \mathcal{F} \subset \Pi^+$, $\langle w, \Sigma\bar{w} \rangle \geq 0$ since this is true in Π^+ by (4.36). Note that $\lambda > 0$ once again by (4.36) so $M \succ 0$ and therefore $\mathcal{F}$ is closed and convex as it is the intersection of a closed and nondegenerate hyperbolic cone with the regular simplex Π^+. The rest of the claim follows from the fact that w_{evw} lies on the boundary of $\mathcal{F}$ and in the interior of Π^+.

Step 2. As $\forall w \in \Pi^+$, $\text{DR}(w^*) \geq \text{DR}(w)$, $\mathcal{F} \subset \tilde{\mathcal{F}}$. To establish $\tilde{\mathcal{F}} \subsetneq \Pi^+$, let us prove that $O_1 := \{w \succ 0, \varrho(w, w_{\text{evw}})\text{DR}(w^*) < \text{DR}(w_{\text{evw}})\}$ is not empty. Considering (4.29) we are tempted to take the minimum on both sides and to do so let $w^\sharp \in \operatorname{argmin}_{w \in \Pi^+} \varrho(w, w_{\text{evw}})$ that does not depend on f and that exists as the objective is continuous on the compact Π^+. As $\varrho(w^\sharp, w_{\text{evw}}) \leq \min \rho(w_{\text{evw}})$, we have by (4.29) $f \circ \rho(w^\sharp) \leq f(\text{DR}(w_{\text{evw}})^{-1} \min \rho(w_{\text{evw}})\mathbf{1})$. If $\min \rho(w_{\text{evw}}) > 0$, then as $\text{DR}(w_{\text{evw}}) > 1$, $f \circ \rho(w^\sharp) < f(\min \rho(w_{\text{evw}})\mathbf{1}) \leq f \circ \rho(w_{\text{evw}})$. Otherwise, we know there exists a ρ-presentative $u \in \mathbb{R}^n \backslash \{0\}$ such that $\min \rho(w_{\text{evw}}) \leq 0 < \min \rho(u)$ and thus $f \circ \rho(w^\sharp) < f(\text{DR}(w_{\text{evw}})^{-1} \min \rho(u)\mathbf{1}) < f \circ \rho(u)$ since $\text{DR}(w_{\text{evw}}) > 1$. All in all, there exists $w^\sharp \in \Pi^+$ such that $f \circ \rho(w^\sharp) < \max_{\mathbb{R}^n \backslash \{0\}} f \circ \rho$, i.e. $w^\sharp$ is not maximally ρ-presentative. Note that one cannot expect $\rho(w^\sharp) \prec \rho(w_{\text{evw}})$ as it contradicts Lemma 4.3. We recall [see (4.31)] that a maximally ρ-presentative w_f is such that $n^{-1}\langle \rho(w_f), \mathbf{1} \rangle \geq \min \rho(w^*)$. Then equipping $\mathbb{R}^n$ with the usual topology, let $A := \{w \in \mathbb{R}^n, w \succ 0\}$, $F := \overline{A}$ its topological closure and the open set $O := \{w \in \mathbb{R}^n \backslash \{0\}, n^{-1}\langle \rho(w), \mathbf{1}\rangle < \min \rho(w^*)\}$. So if $w \in O$ then w cannot be maximally ρ-presentative. We verify that $F \cap O \neq \emptyset$. Using twice Lemma 3.4 and then (3.2),

$$\mathrm{DR}(w_{\mathrm{evw}})\langle \rho(w^\sharp), w_{\mathrm{ew}}\rangle = \varrho(w^\sharp, w_{\mathrm{evw}}) \tag{4.37}$$

$$\leq \min \rho(w_{\mathrm{evw}}) \tag{4.38}$$

$$\leq \langle \phi(w^*), \rho(w_{\mathrm{evw}})\rangle \tag{4.39}$$

$$= \varrho(w_{\mathrm{evw}}, w^*)\mathrm{DR}(w^*)^{-1} \tag{4.40}$$

$$\leq \min \rho(w^*), \tag{4.41}$$

hence $\langle \rho(w^\sharp), w_{\mathrm{ew}}\rangle < \min \rho(w^*)$ which shows that $w^\sharp \in O \cap F$. However by Proposition 5 in Sec. 1 in Chap. 1 of Bourbaki (1961), $O \cap F = O \cap \overline{A} \subset \overline{O \cap A}$ which proves that $w^\sharp \in \overline{O \cap A}$ and that $O_1 = O \cap A \neq \emptyset$. The fact that O_1 is an open set and that $O_1 \subset O$ proves our claim.

To prove that $\tilde{\mathcal{F}}$ is convex let us remark that $\tilde{\mathcal{F}} = \{w \in \Pi^+,\ n^{-1}\langle \rho(w), \mathbf{1}\rangle \geq \min \rho(w^*)\} = \{w \in \Pi^+, n^{-1}\langle \Sigma w, \sigma^{-1}\rangle \geq \sigma(w) \min \rho(w^*)\}$ which is the intersection of Π^+ with a sublevel of a convex function, hence convex. To finish, let us now consider the equality case in the definition of $\tilde{\mathcal{F}}$. For any $w_f \in \mathcal{R}$, it can equivalently be written $\langle n^{-1}\mathbf{1}, \rho(w_f)\rangle = \min \rho(w^*)$. In this case, (4.30) implies $f \circ \rho(w_f) = f \circ \rho(w^*)$ and as w_f is unique $w_f = w^*$ and thus $\langle n^{-1}\mathbf{1}, \rho(w^*)\rangle = \min \rho(w^*)$ which implies in turn $w_f = w^* = \bar{w}$. □

4.3. *Further comments on maximally ρ-presentative portfolios*

Before closing this section, let us make few remarks regarding the concept of maximally ρ-presentative portfolios. First, the symmetry of a function f that yields such a portfolio is important. Indeed, for any measure of exposure considered, there is no reason for the aggregated exposure of any portfolio to depend on the ordering of the assets. Now, without this assumption, it can be noted that any long-only portfolio θ would solve (4.1) using the increasing and linear function $f = \langle \phi(\theta), \cdot\rangle$.

Second, a parallel can be drawn between the mean-variance utility criterion used for portfolio construction (Markowitz 1952) and the objective maximized in this section. Indeed, the function f being increasing by assumption, it will tend to favor portfolios with a higher average exposure. Also, as f is symmetric and concave, it is Schur concave [see Marshall *et al.* (1979)]. Therefore, for portfolios having a given average exposure, those that have exposures that are "less spread out" [in the words of Marshall *et al.* (1979)] will be favored.

In a nutshell, one could view each f generating a maximally ρ-presentative portfolio as providing a particular trade-off between the average and the dispersion of the spectrum of a portfolio. To illustrate this idea, we define $\mathbb{E}(v)$ and $\mathbb{V}ar(v)$ to be the mean and variance of $v \in \mathbb{R}^n$ and remark that

$$f(\rho(w)) = \mathbb{E}(\rho(w)) - \frac{\lambda}{2}\mathbb{V}\mathrm{ar}(\rho(w)) \tag{4.42}$$

is in fact an objective that satisfies Definition 4.1 for $\lambda \in [0, 1)$. Indeed, f is concave symmetric if $\lambda \geq 0$ (hence Schur concave) but increasing only for $\lambda < 1$. Remark that the rightmost term could also be modified to take into account interactions between exposures.

Example 4.9. We conclude this section with Figs. 3 and 4 where we depict the sets of maximally ρ-presentative portfolios $\mathcal{R}$ that we got for, respectively, three and four assets whose covariance matrices are

$$\Sigma_1 = \begin{pmatrix} 1 & -0.4 & -0.8 \\ -0.4 & 1 & 0.7 \\ -0.8 & 0.7 & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & 0 & -0.7 & 0.2 \\ 0 & 1 & -0.3 & -0.6 \\ -0.7 & -0.3 & 1 & 0.5 \\ 0.2 & -0.6 & 0.5 & 1 \end{pmatrix}. \tag{4.43}$$

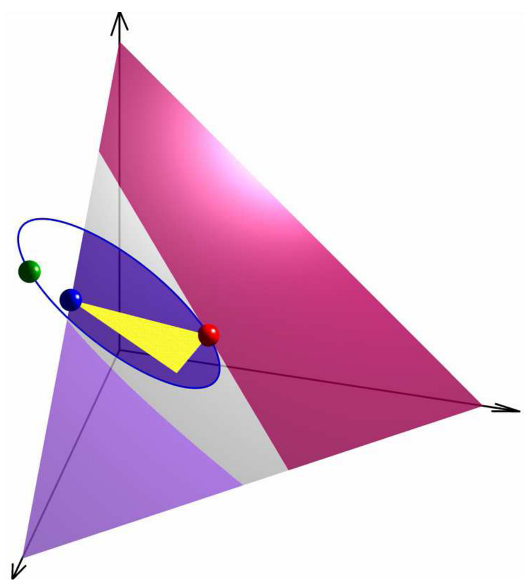

Fig. 3. For $n = 3$ assets, we represent the regular simplex Π^+ along with the set of maximally ρ-presentative portfolios $\mathcal{R}$ (in yellow), the sets $\mathcal{F}$ (in dark violet) and $\tilde{\mathcal{F}}$ (whose complement in Π^+ is indicated in light violet). From left to right the bullets depict the long-short MDP, MDP and EVW with the latter two being maximally ρ-presentative as we are going to see. Note that the long-short MDP and EVW lie on the boundary of the ellipsoid that determines $\mathcal{F}$. The tangent hyperplane to $\mathcal{F}$ at w_{evw} separates Π^+ into two sets such that one of them (depicted in pink) does not contain any maximally ρ-presentative portfolio.

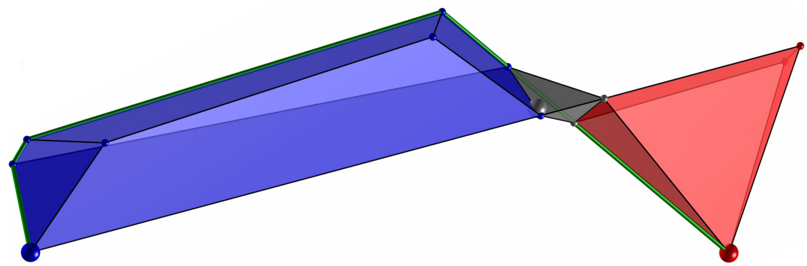

Fig. 4. For $n = 4$ assets, we represent the set of maximally ρ-presentative portfolios $\{(w_1, w_2, w_3) \in [0,1)^3/(w_1, w_2, w_3, 1 - w_1 - w_2 - w_3) \in \mathcal{R}\}$ that is the union of polytopes. From left to right the large bullets depict the MDP, ERC and EVW that are indeed maximally ρ-presentative as we are going to see. In Sec. 5.2.1, we shall also see that the constrained MDPs define a continuous path (depicted in green) within $\mathcal{R}$ that connects the EVW and the MDP.

5. A Framework for Constructing Alternative Strategies

In this section, we investigate whether well-known long-only portfolios such as the EW, EVW and ERC, as well as the MV and MDP with maximum weight constraints, are maximally ρ-presentative. Doing so, we find alternative definitions of these portfolios as maximizers of *basic unconstrained objectives*. We obtain a unifying framework for constructing portfolios as a result, and identify in the case of the constrained MV and MDP the impact of maximum weight constraints on their unconstrained objectives.

5.1. *Alternative definitions of well-known portfolios*

As shown in Sec. 3, the MV, MDP and ERC are ρ-presentative. We investigate in this subsection whether these portfolios but also the EW and the EVW are maximally ρ-presentative.

5.1.1. *The equal volatility-weighted portfolio*

To improve the overall exposure of a portfolio one may maximize the average of its correlations to the assets.

Proposition 5.1. *The EVW is maximally ρ-presentative as it is the unlevered portfolio that maximizes its average correlation to all the assets amongst all nonzero long-short portfolios. Said otherwise,*

$$w_{\text{evw}} = \underset{w \in \Pi}{\operatorname{argmax}} \langle \rho(w), \mathbf{1} \rangle. \tag{5.1}$$

Proof. For $w \in \Pi$, $\langle \mathbf{1}, \rho(w) \rangle = \sigma_\Sigma(w)^{-1} \langle 1 \oslash \sigma, \Sigma w \rangle = \sigma_\Sigma(\mathbf{1} \oslash \sigma) \varrho(w, w_{\text{evw}})$ that is maximized by w_{evw} and any other such unlevered portfolio is perfectly correlated to it and thus identical by Proposition 1.1. $\square$

The fact that the optimum over long-short portfolios is attained by a unique long-only portfolio can also be derived from Lemma 4.3 and Proposition 4.2 without exhibiting the solution. Moreover, let us recall that even though the EVW is maximally ρ-presentative it is not necessarily ρ-presentative, as was shown in Proposition 3.3. However, as expected, it is weakly ρ-presentative as $\langle \mathbf{1}, \rho(w_{\text{evw}})\rangle = \sigma_\Sigma(\mathbf{1} \oslash \sigma) > 0$.

5.1.2. *The most diversified portfolio*

We may wonder whether it is possible to build a portfolio that is both ρ-presentative and maximally ρ-presentative. For a positive answer, let us focus on portfolios that maximize their minimal exposure.

Proposition 5.2. *The MDP w^* is the unlevered portfolio that maximizes its minimal correlation to all assets amongst all long-short portfolios. Moreover, amongst long-short portfolios, the MDP is the unlevered portfolio that maximizes the minimum correlation to any long-only portfolio. Said otherwise,*

$$\operatorname*{argmax}_{w\in\Pi^+} \mathrm{DR}(w) = \operatorname*{argmax}_{w\in\Pi} \min \rho(w) = \operatorname*{argmax}_{w\in\Pi} \min_{\theta\in\Pi^+} \varrho(w,\theta). \tag{5.2}$$

In fact, $\forall (y,w) \in (\mathbb{R}^n\backslash\{0\}) \times \Pi^+$,

$$\min \rho(y) \le \min \rho(w^*) = \min_{\theta\in\Pi^+} \varrho(w^*,\theta) = \mathrm{DR}(w^*)^{-1} \le \mathrm{DR}(w)^{-1}. \tag{5.3}$$

In addition to being ρ-presentative, the MDP is also maximally ρ-presentative.

Proof. We start with the first claim of the proposition. Let $w \in \mathbb{R}^n\backslash\{0\}$ then given that $\phi(w^*) \in \Pi^+$,

$$\min \rho(w) \le \langle \phi(w^*), \rho(w)\rangle = \mathrm{DR}(w^*)^{-1} \varrho(w,w^*) = \min \rho(w^*)\varrho(w,w^*), \tag{5.4}$$

where we used the identities (3.4) and (3.2). Taking the supremum on Π on both ends proves that it is attained by w^* and any other such portfolio $y^* \in \Pi$ satisfies $\min \rho(w^*) = \min \rho(y^*) \le \min \rho(w^*)\varrho(y^*,w^*)$ and is thus perfectly correlated to w^* as $\min \rho(w^*) > 0$ by (3.2), hence $y^* = w^*$ by Proposition 1.1.

It remains to prove the second identity in (5.2). As $\min_{\theta\in\Pi^+} \varrho(w,\theta) \le \min \rho(w)$ with equality for $w = w^*$ by Lemma 3.4, one has $w^* \in \operatorname{argmax}_{w\in\Pi} \min_{\theta\in\Pi^+} \varrho(w,\theta)$. Now for any y^* in the rightmost set,

$$0 < \min \rho(w^*) = \min_{\theta\in\Pi^+} \varrho(w^*,\theta) = \min_{\theta\in\Pi^+} \varrho(y^*,\theta) \le \min \rho(y^*),$$

which proves that y^* is ρ-presentative so by Lemma 3.4 the last inequality is an identity. Then taking $w = y^*$ in (5.4) and simplifying by $\min \rho(y^*) = \min \rho(w^*)$ on both ends, we obtain $\varrho(y^*,w^*) \ge 1$ hence $y^* = w^*$ by Proposition 1.1. $\square$

This proposition proves that, amongst all long-short unlevered portfolios, the MDP is the portfolio that maximizes its minimal exposure to all long-only portfolios.

As such, the *MDP maximizes its lowest exposure to all long-only factors*, defined as factors that are replicable by leveraged long-only portfolios of assets belonging to the universe.

Remark 5.3. In view of these results, one could think of constructing long-only portfolios that minimize their maximal exposure, in the spirit of a minimum variance approach. Formally, one may do so by solving

$$\min_{w\in\Pi^+} \max \rho(w). \tag{5.5}$$

This problem that we already encountered in Proposition 3.5 may admit many local minima and not necessarily a unique global solution. This makes this approach challenging when reaching the implementation phase in a financial setting. Furthermore, the set of optima of the min–max problem may not contain the solution of the max–min problem. One may verify numerically both of these remarks by considering three assets with $\Sigma = C$, $C_{1,2} = C_{1,3} \geq 0.7$, $C_{2,3} < 0.4$ and $C \succ 0$.

5.1.3. *The equal risk contribution portfolio*

Having considered some basic functions f, we pursue with the natural logarithm to prove that the ERC is maximally ρ-presentative.

Proposition 5.4. *The ERC is maximally ρ-presentative since*

$$w_{\text{erc}} = \underset{w\in\Pi}{\operatorname{argmax}} \langle \ln(\rho(w)), \mathbf{1}\rangle, \tag{5.6}$$

where the natural logarithm is taken entry-wise with the convention $\ln \equiv -\infty$ *on* $[-1, 0]$.
Furthermore,

$$\mathrm{DR}(w_{\text{evw}}) \leq \mathrm{DR}(w_{\text{erc}}) \leq \mathrm{DR}(w^*) \quad \text{and} \quad \varrho(w_{\text{erc}}, w_{\text{evw}}) \geq \frac{\mathrm{DR}(w_{\text{evw}})}{\mathrm{DR}(w^*)}. \tag{5.7}$$

Proof. Consider $f : \mathbb{R}^n \to [-\infty, 0]$ defined by $x \mapsto \langle \ln(x), \mathbf{1}\rangle$ with the convention $\ln \equiv -\infty$ on $(-\infty, 0]$. As f admits infinite values its domain is different from $\mathbb{R}^n$ so we need to show that $\sup_\Pi f \circ \rho$ is indeed attained. As there exists a ρ-presentative portfolio $u \in \Pi^+$, there exists $\varepsilon > 0$ such that $\rho(u) > \varepsilon\mathbf{1}$ and thus $\langle \ln(\rho(u)), \mathbf{1}\rangle > n\ln(\varepsilon)$. So we can narrow our search to $\{w \in \Pi, \langle \ln(\rho(w)), \mathbf{1}\rangle \geq n\ln(\varepsilon)/2\}$ which is bounded and closed — by the continuity of $w \mapsto \langle \ln(\rho(w)), \mathbf{1}\rangle$ — and thus compact. This justifies that the sup is attained.

To deal only with finite values in the objective, we can add the nonbinding constraint $\langle \ln(\rho(w)), \mathbf{1}\rangle \geq \ln(\varepsilon)$ in the maximization problem. However as any portfolio w that satisfies this constraint is such that $\prod_{i=1}^n \rho(w)_i \geq \varepsilon$ with $\rho(w)_i \in (0, 1]$, necessarily $\rho(w) \succeq \varepsilon\mathbf{1}$ which in turn implies that $\Sigma w \succeq \sigma_\Sigma(w)\varepsilon\sigma \succeq (\min_\Pi \sigma_\Sigma)\varepsilon\sigma$. The objective remains finite under this new constraint which is less restrictive and not binding either.

Moreover, the optimization is performed over Π and by 0-homogeneity of ρ, this corresponds to three exclusive cases: either $\langle \mathbf{1}, w\rangle = 1$ or $\langle \mathbf{1}, w\rangle = 0$ or $\langle \mathbf{1}, w\rangle = -1$. By Lemma 4.3, given a long-short portfolio there is always a long-only portfolio that improves the objective so we know that any solution is in Π^+ and as a consequence we can discard the two latter nonbinding constraints and keep only $\langle \mathbf{1}, w\rangle = 1$

To sum up, we justified the following identity:

$$\max_{w\in\Pi} \langle \ln(\rho(w)), \mathbf{1}\rangle = \max\{\langle \ln(\rho(w)), \mathbf{1}\rangle / \langle w, \mathbf{1}\rangle = 1,\ \Sigma w \succeq \alpha\sigma\}, \tag{5.8}$$

where we set $\alpha = (\min_\Pi \sigma_\Sigma)\varepsilon$. The objective is finite and continuously differentiable over a set of linear constraints so we may apply the KKT theorem. However, as $\Sigma w \succeq \alpha\sigma$ is not binding, it will not appear in the KKT conditions. Now, if w denotes the maximizer of the latter problem, the KKT condition reads $\nabla_w(w \mapsto \frac{n}{2}\ln(\sigma^2(w)) - \langle \ln(\Sigma w), \mathbf{1}\rangle)(w) = \mu\mathbf{1}$ with $\mu \in \mathbb{R}$. Computing the differential, we get

$$n\Sigma w/\sigma^2(w) = \Sigma(\mathbf{1} \oslash \Sigma w) + \mu\mathbf{1}, \tag{5.9}$$

and then taking the inner product with w we observe that $\mu = 0$. Therefore, composing with Σ^{-1} we get $\Sigma w \odot w = n^{-1}\sigma^2(w)\mathbf{1}$ which, by Corollary 1.2 in Spinu (2013), is solved by a unique portfolio that is the ERC.

Finally, as $w_{\mathrm{erc}} \in \mathcal{R}$, the remaining inequalities follow from Proposition 4.5. □

In this proposition, the inequalities between the Diversification Ratios of the EVW, ERC and MDP are the analogs of those obtained with their volatilities in Appendix A3 of Maillard *et al.* (2010). Furthermore, as noted just before Proposition 4.8, if $\mathbf{1}$ is an eigenvector of the correlation matrix the inequalities of the previous proposition imply that $w_{\mathrm{erc}} = w_{\mathrm{evw}} = w^*$ which is in this case the unique maximally ρ-presentative portfolio.

Now, let us remark that invoking statement (i) in Theorem 4.4, it is clear from the identity

$$\rho(w_{\mathrm{erc}}) = n^{-1}\sigma(w_{\mathrm{erc}}) \oslash (w_{\mathrm{erc}} \odot \sigma) = n^{-1}\mathrm{DR}^{-1}(w_{\mathrm{erc}}) \oslash \phi(w_{\mathrm{erc}}) \tag{5.10}$$

that the ERC is indeed maximally ρ-presentative. However, we went through the effort of the previous analysis to not only give a direct proof of this result but also exhibit a nontrivial objective that is a function of $\rho(w)$. In particular, this objective does not involve explicit long-only constraints as for usual formulations of this problem, which in the absence of such constraints would lead to $2^n - 1$ non long-only solutions as shown in Proposition 1.3 in Spinu (2013). This suggests alternative ways of computing the ERC that could complement the approach taken in that paper.

5.1.4. *The minimum variance portfolio*

In the same spirit as in the previous subsections, we characterize MV portfolios using the spectrum $\rho(w)$.

Proposition 5.5. *One has*

$$\min_{w\in\Pi^+}\sigma(w)=\max_{w\in\mathbb{R}^n\backslash\{0\}}\min(\rho(w)\odot\sigma), \tag{5.11}$$

and the maximum is attained by a unique portfolio (*up to leverage*) *that is the MV. In fact*, $\forall(y,w)\in(\mathbb{R}^n\backslash\{0\})\times\Pi^+$,

$$\min(\rho(y)\odot\sigma)\le\min(\rho(w_{\mathrm{mv}})\odot\sigma)=\sigma(w_{\mathrm{mv}})\le\sigma(w). \tag{5.12}$$

Furthermore, the MV is not necessarily maximally ρ-presentative.

Proof. Let $f:y\in\Pi\mapsto\min(\rho(y)\odot\sigma)$, then if $f(y)\ge\sigma(w_{\mathrm{mv}})$, necessarily $\frac{\Sigma y}{\sigma(y)}\succeq\sigma(w_{\mathrm{mv}})\mathbf{1}$ which implies $\rho(y,w_{\mathrm{mv}})=1$ hence $y=w_{\mathrm{mv}}$. This proves that $\{f\ge\sigma(w_{\mathrm{mv}})\}\subset\{w_{\mathrm{mv}}\}$. To check that the superlevel is not empty we remark that $f(w_{\mathrm{mv}})\ge\sigma(w_{\mathrm{mv}})$. This follows from the KKT theorem applied to $\min_{\Pi^+}\sigma_\Sigma$ that shows that $\exists\lambda\succeq0$ and $\Sigma w_{\mathrm{mv}}/\sigma(w_{\mathrm{mv}})=\sigma(w_{\mathrm{mv}})\mathbf{1}+\lambda$, hence the claim.

Finally, to be maximally ρ-presentative, by Proposition 4.7, the MV needs to be weakly ρ-presentative and to satisfy the bound $\varrho(w_{\mathrm{mv}},w_{\mathrm{evw}})\ge\mathrm{DR}(w_{\mathrm{evw}})/\mathrm{DR}(w^*)$. Consider a situation where all nondiagonal correlations are identical: then $w_{\mathrm{evw}}=w^*$ and as a consequence we also have $w_{\mathrm{mv}}=w^*$. Writing the KKT conditions satisfied by w^* and w_{mv} implies readily that $\mathbf{1}$ is an eigenvector of $CD_\sigma\Sigma^{-1}=CD_\sigma D_\sigma^{-1}C^{-1}D_\sigma^{-1}=D_\sigma^{-1}$ which leads to a contradiction if we consider assets that have different volatilities. □

The previous result also implies that Markowitz's mean-variance portfolios are not necessarily maximally ρ-presentative. To see this, consider a special case of the example provided in the above proof, where correlations between different assets are zero but volatilities are not necessarily identical. In this case, the long-only minimum variance has weights that are proportional to the inverse of the squared volatilities. As these weights are positive for all the assets, this is also the long-short minimum variance portfolio as no long-only constraint is active. Now, if we assume that expected returns of assets are equal, Markowitz's mean-variance portfolios all reduce to the long-only minimum variance portfolio, which is not maximally ρ-presentative as was shown in the proof. The fact that mean-variance efficient portfolios are not always maximally ρ-presentative can also be understood in the context of Choueifaty *et al.* (2013) where it is shown that the MV and EW are not leverage-invariant, as opposed to the ERC, EVW and MDP. More generally, the set of maximally ρ-presentative portfolios is also leverage-invariant as we explain in the introduction of Sec. 6.

5.2. *On the impact of maximum weight constraints*

In practice, asset managers may use maximum weight constraints when imposed by regulators or when using objective functions that are too sensitive to the estimation of their parameters (a common problem for long-short mean-variance portfolios). To address this issue, robust covariance estimators are routinely used by asset managers with some popular choices involving shrinkage methods (Ledoit & Wolf 2003) or factor models (Campbell *et al.* 1997).

The use of maximum weight constraints and robust covariance estimators can be closely related. Indeed, Proposition 1 in Jagannathan & Ma (2003) shows that, for the MV portfolio, imposing nonnegative and maximum weight constraints is equivalent to using a robust version of the original covariance matrix. This matrix is robust in the sense that extreme covariances are most likely to be "shrunk" towards more reasonable values. A limitation of the method is that the modified matrix depends on Lagrange multipliers that are known only after the MV optimization or determined through a numerically demanding maximization of a likelihood function over a set of matrices [cf. Proposition 2 in Jagannathan & Ma (2003)].

Another route proposed in this paper is to identify *a priori* an essentially *unconstrained* optimization problem whose objective depends *explicitly* on the maximum weight constraint, and is equivalent to the original constrained problem. This gives as a result a clear understanding of the impact of the maximum weight constraint. The constrained portfolios we consider here have a *volatility-adjusted maximum weight constraint*, i.e. they belong to

$$\Pi^+_{\sigma,r} := \left\{ w \in \Pi^+ / \forall\, i \in \{1,\ldots,n\}, \frac{w_i\sigma_i}{\langle w,\sigma\rangle} \leq \frac{1}{r} \right\} \tag{5.13}$$

for some real r. In particular, portfolios with *maximum weight constraint* $1/r$ belong to $\Pi^+_{\mathbf{1},r}$.

We first present the unconstrained optimization problems that are equivalent to the original constrained problems solved by the MDP and MV, respectively, and conclude this subsection by discussing the implications of these two results.

5.2.1. *An alternative definition of the constrained most diversified portfolio*

We consider in this subsection an aggregation of the correlation spectrum that generalizes those proposed in Propositions 5.1 and 5.2 for the EVW and the MDP.

Definition 5.6. For $r \in \{1,\ldots,n\}$, the *rank-r ρ-presentativity measure* of $w \in \mathbb{R}^n\backslash\{0\}$, denoted by $\mathrm{RM}_r(w)$, is the average of the r smallest correlations of w to the assets. Considering the reordering $(\rho(w))_{(i)} \leq (\rho(w))_{(i+1)}$,

$$\mathrm{RM}_r(w) := \frac{1}{r}\sum_{i=1}^{r} (\rho(w))_{(i)}. \tag{5.14}$$

Using the lingo of Sec. 4.1, $\mathrm{RM}_r(w) = \langle \rho(w)^{\downarrow}, r^{-1}\mathbf{1}_r\rangle$ where $\mathbf{1}_r$ is the vector whose r first coordinates are equal to one and zero elsewhere. We could also consider real-valued $r \in [1,n]$ thanks to the identity $\langle \rho(w)^{\downarrow}, r^{-1}\mathbf{1}_r\rangle = \min_{\theta\in\Pi^+_{\mathbf{1},r}} \langle \rho(w), \theta\rangle$.

The average of the r smallest elements of a vector is concave, increasing and symmetric. We show that the constrained MDP w^*_r, that maximizes DR over $\Pi^+_{\sigma,r}$, is also the portfolio that maximizes RM_r. This therefore implies that it is maximally ρ-presentative and that it can be obtained by an *unconstrained optimization* of an objective that incorporates the long-only and volatility-adjusted maximum weight constraints.

Proposition 5.7. *The constrained MDP w_r^* is maximally ρ-presentative as it is the unlevered portfolio that maximizes the rank-r ρ-presentativity measure RM_r over nonzero long-short portfolios. Said otherwise,*

$$\operatorname*{argmax}_{w\in\Pi^+_{\sigma,r}} \mathrm{DR}(w) = \operatorname*{argmax}_{w\in\Pi} \mathrm{RM}_r(w). \tag{5.15}$$

In fact, for any $\forall(y, w_r) \in (\mathbb{R}^n\backslash\{0\}) \times \Pi^+_{\sigma,r}$,

$$\mathrm{RM}_r(y) \le \mathrm{RM}_r(w_r^*) = \mathrm{DR}(w_r^*)^{-1} \le \mathrm{DR}(w_r)^{-1}. \tag{5.16}$$

To prove this proposition we shall use properties of the constrained MDP that relate both DR and RM_r.

Proposition 5.8. *w_r^* exists, is unique and $\mathrm{DR}(w_r^*)\mathrm{RM}_r(w_r^*) = 1$. In addition, $\forall(y, w_r) \in (\mathbb{R}^n\backslash\{0\}) \times \Pi^+_{\sigma,r}$,*

$$\mathrm{RM}_r(y) \le \varrho(y, w_r^*)\mathrm{RM}_r(w_r^*), \tag{5.17}$$

$$\mathrm{DR}(w_r) \le \varrho(w_r, w_r^*)\mathrm{DR}(w_r^*). \tag{5.18}$$

Having this proposition at our disposal, we are ready to prove Proposition 5.7.

Proof of Proposition 5.7. The existence of w_r^* follows from Proposition 5.8. Taking the supremum on both sides of (5.17) shows that w_r^* attains it so w_r^* is maximally ρ-presentative, and all portfolios achieving the supremum are perfectly correlated to it. By Proposition 1.1, the MDP is the unique unlevered portfolio that maximizes RM_r. The remaining results follow directly from Proposition 5.8. □

Proof of Proposition 5.8. The function ϕ introduced before Proposition 2.5 is a bijection from $\Pi^+_{\sigma,r} \to \Pi^+_{\mathbf{1},r}$, and $\mathrm{DR}(w) = \sigma_C(\phi(w))^{-1}$. Now as $\Pi^+_{\sigma,r}$ and $\Pi^+_{\mathbf{1},r}$ are compact, and DR and σ_C are continuous on these sets, they reach their extrema and one can write $\phi(\operatorname{argmax}_{\Pi^+_{\sigma,r}} \mathrm{DR}) = \operatorname{argmin}_{\Pi^+_{\mathbf{1},r}} \sigma_C$. Taking x^* in the rightmost set, by Proposition 2.5, we just need to establish $\frac{1}{r}\sum_{i=1}^{r}(\rho_C(x^*))_{(i)} = \sigma_C(x^*)$ to prove the first claim. To do so, the idea is to find the average of the r smallest entries of Cx^* by applying the KKT theorem to $\min_{\Pi^+_{\mathbf{1},r}} \sigma_C$, which, as $C \succeq 0$, implies that there exist $\lambda \succeq 0,\ \mu \succeq 0$ such that $x^* \in \Pi^+_{\mathbf{1},r}$ verifies the KKT conditions

$$Cx^* = s\mathbf{1} + \lambda - \mu, \quad \lambda \odot x^* = 0 \quad \text{and} \quad \mu \odot (r^{-1}\mathbf{1} - x^*) = 0. \tag{5.19}$$

On the one hand, these conditions imply that $\sigma_C^2(x^*) = s - \langle \mu, x^*\rangle$ and $\langle \mu, x^*\rangle = r^{-1}\langle \mathbf{1}, \mu\rangle$, so $s - r^{-1}\langle \mathbf{1}, \mu\rangle = \sigma_C^2(x^*)$. On the other hand, the two last KKT conditions yield three mutually exclusive cases:

$$\begin{cases} x_i^* = 0 & \Rightarrow (\lambda_i \geq 0 \text{ and } \mu_i = 0) \Rightarrow \lambda_i - \mu_i \geq 0 \quad \text{(Case 1)}, \\ 0 < x_i^* < r^{-1} & \Rightarrow (\lambda_i = 0 \text{ and } \mu_i = 0) \Rightarrow \lambda_i - \mu_i = 0 \quad \text{(Case 2)}, \\ x_i^* = r^{-1} & \Rightarrow (\lambda_i = 0 \text{ and } \mu_i \geq 0) \Rightarrow \lambda_i - \mu_i \leq 0 \quad \text{(Case 3)}. \end{cases} \tag{5.20}$$

As $x^* \in \Pi^+_{\mathbf{1},r}$, $\#\{x_i^* > 0\} \geq r$, so the sum of the r smallest entries of $\lambda - \mu$ is obtained through the summation of all the elements of $-\mu$ only (Cases 2 and 3). Therefore,

$$\frac{\sigma_C(x^*)}{r} \sum_{i=1}^{r} (\rho_C(x^*))_{(i)} = \frac{1}{r} \sum_{i=1}^{r} (Cx^*)_{(i)} \tag{5.21}$$

$$= \frac{1}{r} \sum_{i=1}^{r} (s\mathbf{1} + \lambda - \mu)_{(i)} \tag{5.22}$$

$$= s + \frac{1}{r} \sum_{i=1}^{r} (-\mu)_{(i)} \tag{5.23}$$

$$= s - r^{-1} \langle \mathbf{1}, \mu \rangle \tag{5.24}$$

$$= \sigma_C^2(x^*), \tag{5.25}$$

which proves that $\mathrm{DR}(w_r^*)\mathrm{RM}_r(w_r^*) = 1$. To finish, as $C \succ 0$, uniqueness of w_r^* comes from that of x^*.

Now let us turn to the proof of (5.17) and (5.18): by (3.4), for any two portfolios $(y, w_r) \in (\mathbb{R}^n \backslash \{0\}) \times \Pi^+_{\sigma,r}$,

$$\varrho(y, w_r)\mathrm{DR}(w_r)^{-1} = \langle \phi(w_r), \rho(y) \rangle \geq \min_{\theta \in \Pi^+_{\mathbf{1},r}} \langle \theta, \rho(y) \rangle = \mathrm{RM}_r(y). \tag{5.26}$$

Using $\mathrm{RM}_r(w_r^*)\mathrm{DR}(w_r^*) = 1$, the two inequalities follow if we take in turn $w_r = w_r^*$ and then $y = w_r^*$. ◻

On the practical side, this proposition provides the "duality gap" (5.16) which makes it possible to assess the optimality of a long-only portfolio in terms of DR without computing the MDP. Indeed for any $w_r \in \Pi^+_{\sigma,r}$,

$$0 \leq \mathrm{DR}(w_r)^{-1} - \mathrm{DR}(w_r^*)^{-1} \leq \mathrm{DR}(w_r)^{-1} - \mathrm{RM}_r(w_r), \tag{5.27}$$

where on the right-hand side we do not use w_r^*. This can also be useful in an algorithm as a stopping criterion.

Remark 5.9.

(i) We may conclude that w_r^* is maximally ρ-presentative by invoking Theorem 4.4 once the identity $\mathrm{DR}(w_r^*)\mathrm{RM}_r(w_r^*) = 1$ is established.

(ii) For long-only portfolios neither of the two inequalities (5.17) and (5.18) is superior to the other. Indeed, consider three assets with $\Sigma = C$, $C_{1,2} = C_{1,3} = 0.7$, $C_{2,3} = 0.3$ and $r = 1$. Then the sign of

$$\frac{\mathrm{DR}(w)}{\mathrm{DR}(w_r^*)} - \frac{\mathrm{RM}_r(w)}{\mathrm{RM}_r(w_r^*)} = \frac{\sigma(w^*)}{\sigma(w)} - \frac{\min(\rho(w))}{\sigma(w^*)} \tag{5.28}$$

flips when picking $w \in \{e_1, e_2\}$. However (5.17) is more general as it holds for long-short portfolios.

(iii) Moving to another topic, assuming that Σ is positive semi-definite is enough to derive the KKT conditions in the proof of Proposition 5.8. Under this weaker hypothesis, (5.15) can be established along the same lines as an identity between sets. The definiteness comes into play to prove that the MDP is unique and that it is the unique portfolio that maximizes RM_r by Proposition 1.1. Dropping the definiteness of Σ, we still have that all portfolios in the maximizing sets are perfectly correlated. One has to be careful and select $w \in \Pi \backslash \operatorname{Ker}(\Sigma)$ to avoid dividing by zero in the definition of $\rho(w)$. From the beginning, one could have actually balanced the definition of Π and the class of matrices that are allowed by picking them in $\{\Sigma \succeq 0/\sigma_\Sigma > 0 \text{ on } \Pi\}$. This remark is to be related with the concepts of long-only eigenvalue and long-only condition number introduced in Sec. 6 that illustrates that in practice we could consider merely semi-definite covariance matrices for problems involving long-only constraints (see for instance Fig. 5).

5.2.2. *An alternative definition of the constrained minimum variance*

In this subsection, we state a generic result that yields Proposition 5.7 in the special case $\Sigma = C$ and that is obtained along the same lines.

Theorem 5.10. *The minimization of a positive-definite quadratic form over the simplex subject to a uniform maximum constraint can be expressed as an unconstrained optimization as follows*:

$$\min_{w \in \Pi_{1,r}^+} \sigma(w) = \max_{w \in \mathbb{R}^n \backslash \{0\}} \frac{1}{r} \sum_{i=1}^{r} (\rho(w) \odot \sigma)_{(i)}, \tag{5.29}$$

where the maximum is attained by a unique unlevered portfolio that is the constrained MV $w_{\mathrm{mv},r}$ (the EW if $r = n$). In fact, for any $(y, w) \in (\mathbb{R}^n \backslash \{0\}) \times \Pi_{\mathbf{1},r}^+$,

$$\frac{1}{r} \sum_{i=1}^{r} (\rho(y) \odot \sigma)_{(i)} \leq \frac{1}{r} \sum_{i=1}^{r} (\rho(w_{\mathrm{mv},r}) \odot \sigma)_{(i)} = \sigma(w_{\mathrm{mv},r}) \leq \sigma(w). \tag{5.30}$$

This gap can be used to assess the optimality of a portfolio without computing the constrained MV.

Finally, the constrained MV $w_{\mathrm{mv},r}$ (the EW if $r = n$) is not necessarily maximally ρ-presentative.

Proof. Conducting an analysis similar to the previous subsection, one obtains the first two assertions. It remains to prove that the constrained MV is not necessarily maximally ρ-presentative. The case $r = 1$ that corresponds to the unconstrained MV was handled in the proof of Proposition 5.5. To be maximally ρ-presentative, by Proposition 4.7, the constrained MV needs to be weakly ρ-presentative and to satisfy the bound $\varrho(w_{\mathrm{mv},r}, w_{\mathrm{evw}}) \geq \mathrm{DR}(w_{\mathrm{evw}})/\mathrm{DR}(w^*)$. Consider a situation where all correlations are identical. Then $w_{\mathrm{evw}} = w^*$ hence $w_{\mathrm{mv},r} = w_{\mathrm{evw}}$. Now, if $r > 1$, then by sending the volatility of a single asset to zero, its weight in the EVW can be made as close to one as one wishes. In this situation, the constrained MV whose weights are bounded by $1/r$ is in general different from the EVW. Taking $r = n$ proves that EW is not maximally ρ-presentative. $\square$

This result shows that the constrained MV or the EW maximizes an aggregated exposure, where individual exposures are given by $\rho(w) \odot \sigma$. These are usually called *marginal risk contributions* [see Roncalli (2013)]. As such, using $\rho(w) \odot \sigma$ as an alternative measure of exposure would lead to a new framework, where these portfolios would indeed be maximally exposed. Conducting an analysis similar to the proof of Theorem 4.4, we can prove that the set of maximally exposed portfolios given this measure of exposures is exactly

$$\mathcal{R}^\sigma := \{w \in \Pi^+, \sigma(w)^2 = \langle w^\uparrow, (\Sigma w)^\downarrow \rangle\}, \tag{5.31}$$

which is small in the sense of Theorem 4.4 and for any $w \in \mathcal{R}^\sigma$, $\varrho(w, w_{\mathrm{ew}}) \geq \sigma(w)/\sigma(w_{\mathrm{ew}})$. In a similar way, one can carry the results of Theorem 4.4 to the set $\mathcal{R}^\mu$ that is associated to a general-weighted measure $\rho(w) \odot \mu$ with $\mu \succ 0$. In addition to our discussion before Example 4.9, this offers another alternative to the celebrated approach of Markowitz and is left for further research.

5.2.3. *Implications of these alternative definitions*

The results obtained in Secs. 5.2.1 and 5.2.2 allow to identify *a priori* how the objectives maximized by the MDP and MV are modified by the addition of maximum weight constraints, that are volatility-adjusted for the MDP. Consider for example the case of the MV portfolio in a universe of 500 assets. Theorem 5.10 shows that minimizing the volatility of a long-only portfolio is equivalent to maximizing the minimal marginal risk contribution of a long-short portfolio with weights summing to one. Moreover, if a maximum weight constraint of 2% is added, the problem becomes equivalent to the maximization of the average of the lowest 50 marginal risk contributions of such a long-short portfolio.

A related result is provided in Proposition 1 in Jagannathan & Ma (2003), whereby the problem of minimizing the volatility of a long-short portfolio whose weights sum to one is studied. The authors show that adding minimum and maximum constraints to this problem is equivalent to solving the original problem using a modified covariance matrix that is clearly identified. Nevertheless, its analytical form is *a priori* unknown as it depends on the Lagrange multipliers associated to the added constraints. However, the authors provide an interpretation of this modified matrix, and show that the adjustment brought to the original matrix "can reduce sampling error". In the remaining of Jagannathan & Ma (2003), an empirical study is conducted that confirms these claims.

A connection between the results provided in Jagannathan & Ma (2003) and both Proposition 5.7 and Theorem 5.10 can easily be made in a context where the covariance matrix of the assets needs to be estimated. In this case, the correlation spectrum $\rho(w)$ and the marginal risk contributions $\rho(w)\odot\sigma$ are subject to estimation errors. Coming back to our MV example, this means that adding a 2% maximum weight constraint is equivalent to maximizing an objective that now averages 50 estimated variables. This arguably can contribute to "reduce sampling error" which can of course come at a cost and introduce a bias. Any further statistical analysis is beyond the scope of this paper and is left for future research.

However, a first step toward such a statistical study is proposed in Sec. 6 where we study the impact of deterministic variations of the covariances on the set of maximally ρ-presentative portfolios. This analysis gives another path to understand how the introduction of long-only constraints may stabilize portfolios that also result from an optimization of the spectrum — as is the case of the long-only minimum variance portfolio. Indeed, we show that, for such problems, long-only constraints are manifested by estimates on the discrepancy between portfolio sets that involve the analog of "long-only condition numbers". The latter are no greater than the usual condition numbers and may remain finite for non-invertible covariance matrices. This ensures that, even in such an unfavorable situation, optimized portfolios may remain closely correlated for small variations of the covariances. This result gives additional clues as to why the introduction of constraints may "reduce sampling noise" as observed in Jagannathan & Ma (2003).

5.3. *A unifying framework*

So far we have shown that many well-known — possibly constrained — investment strategies maximize their overall exposure to the assets, as measured by some real-valued f. This in fact provides a unifying framework, whereby all strategies maximize an unconstrained objective that is a function of the spectrum $\rho(w)$.

We summarize most of these results in Table 1. Given an investment strategy that is indicated in the first column, the second column provides the *primal objective* that is maximized by the corresponding portfolio, while the third column contains its well-known *dual definition*. The use of the primal–dual terminology is justified in Sec. 7.2. The following columns then indicate whether the considered portfolio is always long-only, ρ-presentative or maximally ρ-presentative. Key remarks and references are indicated in the last column.

We have also included in the table three portfolios that are obtained using functions f that do not satisfy at least one of the three assumptions of Definition 4.1.

Table 1. An alternative framework for constructing portfolios. We used the following abbreviations: constr: *constrained*, eigv: *eigenvector*, LO: *long-only*, LS: *long-short*, ρ-pr: *ρ-presentative* and max ρ-pr: *maximally ρ-presentative*. Let δ_{Π^+} denote the function that vanishes on Π^+ and that is $+\infty$ elsewhere. Mean-var ρ (respectively, Min max ρ) was defined by Eq. (4.42) (respectively, Eq. (5.5))

Investment strategy name	Primal approach: Portfolios maximize $f \circ \rho(w) =$	Dual approach: Weights proportional to	LO	ρ-pr	max ρ-pr	Remarks and References
EW	$\langle \rho(w) \odot \sigma, \mathbf{1} \rangle$	$\mathbf{1}$	$\times$			cf. Prop. 3.3 and Thm. 5.10
EVW	$\langle \rho(w), \mathbf{1} \rangle$	$\mathbf{1} \oslash \sigma$	$\times$		$\times$	cf. Props. 3.3 and 5.1
Generic LO	$\langle \rho(w), \phi(\theta) \rangle$	$\theta \in \Pi^+ \backslash \{w_{\text{evw}}\}$	$\times$			f not symmetric, cf. Sec. 4.3
ERC	$\langle \ln(\rho(w)), \mathbf{1} \rangle$	$w_i(\Sigma w)_i = \frac{\sigma^2(w)}{n}$	$\times$	$\times$	$\times$	cf. Props. 3.3 and 5.4
MV	$\min \rho(w) \odot \sigma$	$\operatorname{argmin}_{\Pi^+} \sigma_\Sigma$	$\times$	$\times$		f not symmetric, cf. Props. 3.3 and 5.5
MDP	$\min \rho(w)$	$\operatorname{argmax}_{\Pi^+} \mathrm{DR}$	$\times$	$\times$	$\times$	cf. Props. 3.3 and 5.2
Constr MV	$\sum_{i=1}^{r} (\rho(w) \odot \sigma)_{(i)}$	$\operatorname{argmin}_{\Pi^+_{\mathbf{1},r}} \sigma_\Sigma$	$\times$			f not symmetric, cf. Thm. 5.10
Constr MDP	$\sum_{i=1}^{r} (\rho(w))_{(i)}$	$\operatorname{argmax}_{\Pi^+_{\sigma,r}} \mathrm{DR}$	$\times$		$\times$	cf. Prop. 5.7
LS MDP	$-\mathbb{V}\mathrm{ar}(\rho(w))$	$\pm\Sigma^{-1}\sigma$		$\times$		Long–short and ρ-pr, cf. Prop. 7.3
Assets	$\langle \rho(w)^p, \mathbf{1} \rangle, p > 2$	any e_i if $\Sigma = I$	$\times$			f not concave, several maximizers
First eigv of Σ	$\Vert \rho(w) \odot \sigma \Vert_2$	$\operatorname{argmax}_w \frac{\sigma(w)}{\Vert w \Vert_2}$				f not concave, first PCA factor of Σ
Max ρ-pr θ	$\langle \rho(w)^{\downarrow}, \phi(\theta)^{\uparrow} \rangle$	$\theta \in \mathcal{R}$	$\times$		$\times$	cf. Thm. 4.4
Mean-var ρ	$\mathbb{E}(\rho(w)) - \frac{\lambda}{2}\mathbb{V}\mathrm{ar}(\rho(w))$		$\times$		$\times$	$\lambda \in [0,1)$, cf. Section 4.3
$w^\sharp$	$-\langle \rho(w), \mathbf{1} \rangle - \delta_{\Pi^+}(w)$		$\times$			never max ρ-pr, proof of Prop. 4.8
Min max ρ	$-\max \rho(w) - \delta_{\Pi^+}(w)$		$\times$			not unique, Rmk. 5.3

The first such portfolio is the first eigenvector of the covariance matrix Σ. It is obtained with $f \circ \rho(w) = \Vert \rho(w) \odot \sigma \Vert$. Indeed,

$$\max_{w \in \mathbb{R}^n \backslash \{0\}} \Vert \rho(w) \odot \sigma \Vert^2 = \max_{w \in \mathbb{R}^n \backslash \{0\}} \frac{\langle \Sigma w, \Sigma w \rangle}{\langle \Sigma w, w \rangle} = \max_{w \in \mathbb{R}^n \backslash \{0\}} \frac{\Vert w \Vert^2}{\langle \Sigma^{-1} w, w \rangle}, \tag{5.32}$$

which is solved by the eigenvector of Σ associated to its largest eigenvalue. If volatilities are not identical, this function does not satisfy any of our assumptions and the resulting portfolio maximizes an aggregation of its *absolute exposures* rather than its exposures. Second, we called "Assets" the portfolios reduced to single assets that are obtained by maximizing $f(x) = \sum_{i=1}^n x_i^p$ when we specialize $\Sigma = I$ and consider $p \in (2, +\infty]$. Indeed, for any such p and any $x \in \mathbb{R}^n \backslash \{0\}$, $f(x) \leq \|x\|_p^p \leq \|x\|_2^p$ with equality between these terms occurring only at elements of the canonical basis. This shows that multiple solutions may be obtained when using a function f that is not concave. Finally, "generic LO" is a generic long-only portfolio that is different from the EVW portfolio and is obtained with a function f that depends on the portfolio weights and is thus not symmetric.

In Table 1, all primal objectives use basic functions of $\rho(w)$ and we could also think of using convex combinations of these objectives. This would allow to retrieve all the intermediary portfolios in the spirit of Jurczenko *et al.* (2015) and Richard & Roncalli (2015) using a single objective. Note that if we start with two basic objectives that satisfy Definition 4.1, we can guarantee that the resulting composite objective will also satisfy Definition 4.1 and therefore produce a maximally ρ-presentative portfolio. Using maximum weight constraints is an alternative approach to build such intermediary portfolios as was illustrated in Fig. 4 where we depicted a path connecting the EVW to the MDP. This path resides entirely in the set of maximally ρ-presentative portfolios as proven in Proposition 5.7.

Finally, remark that the correlation of a portfolio to all the assets of its investment universe is easily computed and one does not need to know the weights. As will be shown is Sec. 7.3, this allows us to assess whether a given fund is close to realize an objective in terms of $\rho(w)$ without knowing its holdings.

6. Stability of the Set of Maximally ρ-presentative Portfolios

We now turn to the stability of the set of maximally ρ-presentative portfolios, and investigate how a change in the covariance matrix from Σ to $\tilde{\Sigma}$ (both being positive-definite) may affect the associated sets $\mathcal{R}_\Sigma$ and $\mathcal{R}_{\tilde{\Sigma}}$. In the sequel, let C, $\tilde{C}$ and σ, $\tilde{\sigma}$ denote the relevant correlation matrices and volatilities.

Let us first note that the set of maximally ρ-presentative portfolios is *leverage-invariant* in the sense that whenever only asset volatilities change, a simple rescaling allows to retrieve the new maximally ρ-presentative portfolios [cf. also Choueifaty *et al.* (2013) on the topic]. Indeed, any portfolio $w_\Sigma \in \mathcal{R}_\Sigma$ obtained via the maximization of a function f that satisfies the assumptions of Definition 4.1, i.e. $w_\Sigma = \text{argmax}_{\Pi^+} f \circ \rho_\Sigma$, can be obtained in two steps. Since in Proposition 2.5 we show that for any $w \in \Pi^+$, $\rho_\Sigma(w) = \rho_C(\phi_\sigma(w))$, the first step amounts to solving for $x_C = \text{argmax}_{\Pi^+} f \circ \rho_C$ that depends only on the correlation matrix. The second step amounts to writing $w_\Sigma = \phi_\sigma^{-1}(x_C)$ that depends only on the volatilities. As a result, $\mathcal{R}_{\tilde{\Sigma}} = \phi_{\tilde{\sigma}}^{-1} \circ \phi_\sigma(\mathcal{R}_\Sigma)$. However, if correlations change along with volatilities, the answer is more involved and we dedicate the following proposition to it.

Proposition 6.1. *Suppose that Σ, $\tilde{\Sigma}$ are two symmetric and positive-definite matrices, that C, $\tilde{C}$ and σ, $\tilde{\sigma}$ are their respective correlation matrices and vectors of volatilities and that $\mathcal{R}_\Sigma$, $\mathcal{R}_{\tilde{\Sigma}}$ denote the associated maximally ρ-presentative sets.*

Then, the following statements hold.

(i) *The volume of the symmetric difference between* $\mathcal{R}_\Sigma$ *and* $\mathcal{R}_{\tilde{\Sigma}}$ *tends to zero along with the distance between the matrices, i.e.*

$$\lim_{\tilde{\Sigma}\to\Sigma} \lambda_{n-1}(\mathcal{R}_\Sigma \,\Delta\, \mathcal{R}_{\tilde{\Sigma}}) = 0, \tag{6.1}$$

where the symmetric difference Δ *is taken in* Π^+ *and* λ_{n-1} *denotes the* $(n-1)$-*dimensional Lebesgue measure.*

(ii) *Considering the Euclidean distance between portfolios of* Π^+, *we can control the induced Hausdorff distance* $d_{\mathcal{H}}$ *between* $\mathcal{R}_\Sigma$ *and* $\mathcal{R}_{\tilde{\Sigma}}$ *as a function of the distance between the matrices* Σ *and* $\tilde{\Sigma}$. *More precisely, the mapping* $\Sigma \mapsto \mathcal{R}_\Sigma$ *satisfies the local Hölder condition*

$$d_{\mathcal{H}}(\mathcal{R}_\Sigma, \mathcal{R}_{\tilde{\Sigma}}) \leq c_{\sigma,\tilde{\sigma}}\|\sigma - \tilde{\sigma}\| + c_{\Sigma,\tilde{\Sigma}}\|C - \tilde{C}\|^{\frac{1}{2}}, \tag{6.2}$$

with

$$c_{\sigma,\tilde{\sigma}} \leq n\xi(1+\sqrt{n}\xi)/\min(\min(\sigma),\min(\tilde{\sigma})), \tag{6.3}$$

$$c_{\Sigma,\tilde{\Sigma}} \leq (\xi + \sqrt{n}\xi^2)(\sqrt{n}+n)n^{\frac{1}{4}}\frac{2\sqrt{2}}{\sqrt{\mu_M}}\left(\frac{\mu_M}{\mu_m}\right)^{\frac{3}{4}}, \tag{6.4}$$

where we denoted μ_m (*respectively,* μ_M) *as the smallest* (*respectively, largest*) *of the eigenvalues of* C *and* $\tilde{C}$ *and* $\xi := \max(\max(\sigma)/\min(\sigma), \max(\tilde{\sigma})/\min(\tilde{\sigma}))$ *that can be thought of as the volatility dispersion.*

(iii) *When volatilities are fixed* (*i.e.* $\sigma = \tilde{\sigma}$), *we have*

$$d_{\mathcal{H}}(\mathcal{R}_C, \mathcal{R}_{\tilde{C}}) \leq \frac{2\sqrt{2n}}{\sqrt{\mu_M}}\left(\frac{\mu_M}{\mu_m}\right)^{\frac{3}{4}}\|C - \tilde{C}\|^{\frac{1}{2}}. \tag{6.5}$$

Equipping Π^+ *with the distance* $d_{\varrho_C}(x,y) := \sqrt{2(1-\varrho_C(x,y))}$, *the induced Hausdorff distance* $\tilde{d}_{\mathcal{H}}$ *satisfies*

$$d_{\tilde{\mathcal{H}}}(\mathcal{R}_C, \mathcal{R}_{\tilde{C}}) \leq \frac{2\sqrt{2}}{\sqrt{\mu_M}}\left(\frac{\mu_M}{\mu_m^+}\right)^{\frac{3}{4}}\|C - \tilde{C}\|^{\frac{1}{2}}, \tag{6.6}$$

where we introduce

$$\mu^+_{\min}(C) := \min_{\substack{\|x\|=1\\ x\succeq 0}} \langle Cx, x\rangle, \tag{6.7}$$

the smallest "long-only eigenvalue" of C *and define* $\mu_m^+ := \min(\mu^+_{\min}(C), \mu^+_{\min}(\tilde{C}))$.

To be more specific, for any $x \in \mathcal{R}_C$,

$$\min_{\tilde{x}\in\mathcal{R}_{\tilde{C}}} \|x - \tilde{x}\|^2 \leq 2\left(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|\right)^2 \times \left(1 + \mathrm{DR}(C) + 2\|\tilde{C}^{\frac{1}{2}}\|\mathrm{DR}(C)\right)\|C - \tilde{C}\|, \tag{6.8}$$

$$\min_{\tilde{x}\in\mathcal{R}_{\tilde{C}}} \sigma_C(x-\tilde{x})^2 \le 2\left(1+\|C^{-\frac{1}{2}}\mathbf{1}\|^2\right) \times (1+\mathrm{DR}(C))\left(1+\|\tilde{C}^{\frac{1}{2}}\|\mathrm{DR}(C)\right)\|C-\tilde{C}\|, \tag{6.9}$$

$$\min_{\tilde{x}\in\mathcal{R}_{\tilde{C}}} (1-\varrho_C(x,\tilde{x})) \le \frac{2}{\mu^+_{\min}(C)}\left(1+\frac{\mu_{\max}(\tilde{C})^{\frac{1}{2}}}{\mu^+_{\min}(\tilde{C})^{\frac{1}{2}}+\mu^+_{\min}(C)^{\frac{1}{2}}}\right)\|C-\tilde{C}\|, \tag{6.10}$$

where, abusing notations, we define $\mathrm{DR}(C) := \max_{\Pi^+}\mathrm{DR}_C$ *to be the maximum* DR *reached by a long-only portfolio formed over the universe of assets whose correlation is* C. *Note that one always has* $\mu_{\max}(\tilde{C})^{\frac{1}{2}} = \|\tilde{C}^{\frac{1}{2}}\| \le \sqrt{n}$, *which allows to eliminate* $\tilde{C}$ *in the local constants of the previous inequalities.*

Proof of (i). As in the proof of Theorem 4.4, let $\Delta_p := \{w \in \Pi^+/p\circ\phi_\sigma(w) = \phi_\sigma(w)^\uparrow, p\circ\rho(w) = \rho(w)^\downarrow\}$ and let $\tilde{\Delta}_p$ be its counterpart associated to $\tilde{\Sigma}$ and recall that $\mathcal{R}_\Sigma = \bigcup_{p\in\mathfrak{S}_n}\Delta_p$ and $\mathcal{R}_{\tilde{\Sigma}} = \bigcup_{p\in\mathfrak{S}_n}\tilde{\Delta}_p$. Then we have

$$\begin{aligned}
\lambda_{n-1}(\mathcal{R}_\Sigma\,\Delta\,\mathcal{R}_{\tilde{\Sigma}}) &= \lambda_{n-1}[(\mathcal{R}_\Sigma\cap(\Pi^+\backslash\mathcal{R}_{\tilde{\Sigma}}))\cup(\mathcal{R}_{\tilde{\Sigma}}\cap(\Pi^+\backslash\mathcal{R}_\Sigma))] &(6.11)\\
&= \lambda_{n-1}[\mathcal{R}_\Sigma\cap(\Pi^+\backslash\mathcal{R}_{\tilde{\Sigma}})]+\lambda_{n-1}[\mathcal{R}_{\tilde{\Sigma}}\cap(\Pi^+\backslash\mathcal{R}_\Sigma)] &(6.12)\\
&= \lambda_{n-1}\left[\bigcup_{p\in\mathfrak{S}_n}\left(\Delta_p\cap\left(\bigcap_{q\in\mathfrak{S}_n}\Pi^+\backslash\tilde{\Delta}_q\right)\right)\right] &(6.13)\\
&\quad+\lambda_{n-1}\left[\bigcup_{p\in\mathfrak{S}_n}\left(\tilde{\Delta}_p\cap\left(\bigcap_{q\in\mathfrak{S}_n}\Pi^+\backslash\Delta_q\right)\right)\right] &(6.14)\\
&\le \sum_{p\in\mathfrak{S}_n}[\lambda_{n-1}(\Delta_p\cap(\Pi^+\backslash\tilde{\Delta}_p))+\lambda_{n-1}(\tilde{\Delta}_p\cap(\Pi^+\backslash\Delta_p))]. &(6.15)
\end{aligned}$$

For fixed $p\in\mathfrak{S}_n$, we focus on the term $\lambda_{n-1}\big(\Delta_p\cap(\Pi^+\backslash\tilde{\Delta}_p)\big)$ as all the other terms can be treated in a similar way and observe that it is equal to

$$\begin{aligned}
&\lambda_{n-1}(\{w\in\Pi^+/p(\sigma\odot w)=(\sigma\odot w)^\uparrow, p((\Sigma w)\oslash\sigma)=((\Sigma w)\oslash\sigma)^\downarrow\} &(6.16)\\
&\quad\backslash\{w\in\Pi^+/p(\tilde{\sigma}\odot w)=(\tilde{\sigma}\odot w)^\uparrow, p((\tilde{\Sigma}w)\oslash\tilde{\sigma})=((\tilde{\Sigma}w)\oslash\tilde{\sigma})^\downarrow\}) &(6.17)\\
&\le \lambda_{n-1}(\{w\in\Pi^+/p(\sigma\odot w)=(\sigma\odot w)^\uparrow\} &(6.18)\\
&\quad\backslash\{w\in\Pi^+/p(\tilde{\sigma}\odot w)=(\tilde{\sigma}\odot w)^\uparrow\}) &(6.19)\\
&\quad+\lambda_{n-1}(\{w\in\Pi^+/p(\sigma\odot\Sigma w)=(\sigma\odot\Sigma w)^\downarrow\} &(6.20)\\
&\quad\backslash\{w\in\Pi^+/p(\tilde{\sigma}\odot\tilde{\Sigma}w)=(\tilde{\sigma}\odot\tilde{\Sigma}w)^\downarrow\}), &(6.21)
\end{aligned}$$

where the two terms are of the form $\lambda_{n-1}((\Pi^+\cap L\delta_p)\backslash(\Pi^+\cap\tilde{L}\delta_p))$ with $\delta_p := \{p(w)=w^\uparrow\}$ [or $\delta_p := \{p(w)=w^\downarrow\}$] and with L, $\tilde{L}$ being two linear mappings. It remains to prove that these two terms converge to zero. For that it suffices to invoke the dominated convergence theorem since the characteristic functions of the different arguments converge pointwise to zero λ_{n-1}-a.e. $\square$

Proof of (ii). Step 1. Our aim is to estimate the distance between two portfolios $w \in \mathcal{R}_\Sigma$ and $\tilde{w} \in \mathcal{R}_{\tilde{\Sigma}}$ in terms of the distance between the positive-definite and symmetric matrices Σ and $\tilde{\Sigma}$. Let us first observe that

$$\forall w, \tilde{w} \in \Pi^+, \|\phi_\sigma(w) - \phi_{\tilde{\sigma}}(\tilde{w})\| \leq \|\phi_\sigma(w) - \phi_{\tilde{\sigma}}(w)\| + \|\phi_{\tilde{\sigma}}(w) - \phi_{\tilde{\sigma}}(\tilde{w})\|, \tag{6.22}$$

where we can bound the first term as follows:

$$\begin{aligned} \|\phi_\sigma(w) - \phi_{\tilde{\sigma}}(w)\| &\leq \frac{\|w\|\|\sigma - \tilde{\sigma}\|}{\langle w, \sigma\rangle} + \|w \odot \tilde{\sigma}\| \left| \frac{1}{\langle w, \sigma\rangle} - \frac{1}{\langle w, \tilde{\sigma}\rangle} \right| \qquad (6.23)\\ &\leq \frac{1}{\langle w, \sigma\rangle}\left[\|w\|\|\sigma - \tilde{\sigma}\| + \|w\|\|\tilde{\sigma}\| \frac{\|w\|\|\sigma - \tilde{\sigma}\|}{\langle w, \tilde{\sigma}\rangle}\right] \qquad (6.24)\\ &\leq [\min(\sigma)]^{-1}\left(1 + \|\tilde{\sigma}\|[\min(\tilde{\sigma})]^{-1}\right)\|\sigma - \tilde{\sigma}\|, \qquad (6.25)\end{aligned}$$

since $\forall w \in \Pi$, $\|w \odot \sigma\| \leq \|w \odot \sigma\|_1 = \langle |w|, \sigma\rangle \leq \|w\|\|\sigma\| \leq \|w\|_1\|\sigma\| = \|\sigma\|$. For the second term, one has

$$\begin{aligned} \|\phi_{\tilde{\sigma}}(w) - \phi_{\tilde{\sigma}}(\tilde{w})\| &\leq \|\tilde{\sigma}\| \frac{\|w\langle \tilde{w}, \tilde{\sigma}\rangle - \tilde{w}\langle w, \tilde{\sigma}\rangle\|}{\langle w, \tilde{\sigma}\rangle\langle \tilde{w}, \tilde{\sigma}\rangle} \qquad (6.26)\\ &\leq \|\tilde{\sigma}\| \frac{\|w - \tilde{w}\|\langle \tilde{w}, \tilde{\sigma}\rangle + \|\tilde{w}\||\langle w - \tilde{w}, \tilde{\sigma}\rangle|}{\langle w, \tilde{\sigma}\rangle\langle \tilde{w}, \tilde{\sigma}\rangle} \qquad (6.27)\\ &\leq [\min(\tilde{\sigma})]^{-1}(1 + \|\tilde{\sigma}\|[\min(\tilde{\sigma})]^{-1})\|\tilde{\sigma}\|\|w - \tilde{w}\|. \qquad (6.28)\end{aligned}$$

Therefore, we can express the Euclidean distance between w and $\tilde{w}$ in terms of that between x and $\tilde{x}$ since

$$\begin{aligned} \|w - \tilde{w}\| &= \|\phi_{\sigma^{-1}}(x) - \phi_{\tilde{\sigma}^{-1}}(\tilde{x})\| \qquad (6.29)\\ &\leq (1 + \max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|) \qquad (6.30)\\ &\quad \times (\max(\sigma)\|\sigma^{-1}\|\|\tilde{\sigma}^{-1}\|\|\sigma - \tilde{\sigma}\| + \max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|\|x - \tilde{x}\|). \qquad (6.31)\end{aligned}$$

This allows us to assume that $\sigma = \tilde{\sigma} = \mathbf{1}$ in the sequel and continue working only with correlations.

Step 2. To be able to estimate the distance in Π^+ between $\mathcal{R}_C$ and $\mathcal{R}_{\tilde{C}}$, we are first going to relate the Euclidean distance $\|x - \tilde{x}\|$ to $\varrho_C(x, \tilde{x})$ for any $x, \tilde{x} \in \Pi_1 := \{x \in \mathbb{R}^n / \langle x, \mathbf{1}\rangle = 1\}$. We recall that in Proposition 2.2 we exhibited a bijection between Π and $\mathcal{E}$. In a similar way, we can recover any $x \in \Pi_1$ from its spectrum since we have $x = C^{-1}(\sigma_C(x)\rho_C(x))$ with $\sigma_C(x)^{-1} = \langle C^{-1}\mathbf{1}, \rho_C(x)\rangle$. The first identity implies that

$$\|x - \tilde{x}\| \leq \|C^{-\frac{1}{2}}\|\sigma_C(x)\|\rho_C(x) - \rho_C(\tilde{x})\|_{C^{-1}} + |\sigma_C(x) - \sigma_C(\tilde{x})|\sigma_C(\tilde{x})^{-1}\|\tilde{x}\|, \tag{6.32}$$

whereas the second one implies

$$\frac{|\sigma_C(x) - \sigma_C(\tilde{x})|}{\sigma_C(x)\sigma_C(\tilde{x})} = |\sigma_C(x)^{-1} - \sigma_C(\tilde{x})^{-1}| \tag{6.33}$$

$$= |\langle C^{-\frac{1}{2}}\mathbf{1}, C^{-\frac{1}{2}}(\rho_C(x) - \rho_C(\tilde{x}))\rangle| \tag{6.34}$$

$$\leq \|C^{-\frac{1}{2}}\mathbf{1}\|\|\rho_C(x) - \rho_C(\tilde{x})\|_{C^{-1}}, \tag{6.35}$$

which can be combined with

$$\|\rho_C(x) - \rho_C(\tilde{x})\|^2_{C^{-1}} = 2(1 - \langle\rho_C(x), \rho_C(\tilde{x})\rangle_{C^{-1}}) = 2(1 - \varrho_C(x, \tilde{x})). \tag{6.36}$$

All in all, given that $x \in \Pi^+$ and $\tilde{x} \in \Pi^+$ play symmetric roles,

$$\|x - \tilde{x}\|^2 \leq \min(\sigma_C(x)^2, \sigma_C(\tilde{x})^2)\left(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|\right)^2\left(2(1 - \varrho_C(x, \tilde{x}))\right) \tag{6.37}$$

where we used the fact that $\|x\| \leq 1$ and $\|\tilde{x}\| \leq 1$. Similarly, the squared tracking error in terms of C can be dominated since

$$\sigma_C(x - \tilde{x})^2 = (\sigma_C(x) - \sigma_C(\tilde{x}))^2 + 2(1 - \varrho_C(x, \tilde{x}))\sigma_C(x)\sigma_C(\tilde{x}) \tag{6.38}$$

$$\leq (\|C^{-\frac{1}{2}}\mathbf{1}\|^2\sigma_C(x)\sigma_C(\tilde{x}) + 1)\sigma_C(x)\sigma_C(\tilde{x})(2(1 - \varrho_C(x, \tilde{x}))). \tag{6.39}$$

We are ready to express the distance between $\mathcal{R}_C$ and $\mathcal{R}_{\tilde{C}}$ as a function of the distance between the matrices.

Step 3. Here the main idea is to use the surjective mapping $x \in \Pi^+ \mapsto z_x := \operatorname{argmax} f_x \circ \rho_{\tilde{C}} = \operatorname{argmax}_{z\in\Pi}\langle x^\uparrow, \rho_{\tilde{C}}(z)^\downarrow\rangle \in \mathcal{R}_{\tilde{C}}$, that we introduced in Theorem 4.4 and which can be thought of as the "projection" of $x \in \mathcal{R}_\mathcal{C}$ on $\mathcal{R}_{\tilde{C}}$. Let $x \in \mathcal{R}_C$, then

$$\sigma_C(x)\varrho_C(x, z_x) = \langle x, \rho_C(z_x)\rangle \tag{6.40}$$

$$\geq \langle x^\uparrow, \rho_C(z_x)^\downarrow\rangle \tag{6.41}$$

$$= \langle x^\uparrow, \rho_{\tilde{C}}(z_x)^\downarrow\rangle + \langle x^\uparrow, \rho_C(z_x)^\downarrow - \rho_{\tilde{C}}(z_x)^\downarrow\rangle \tag{6.42}$$

$$\geq \langle x^\uparrow, \rho_{\tilde{C}}(x)^\downarrow\rangle + \langle x^\uparrow, \rho_C(z_x)^\downarrow - \rho_{\tilde{C}}(z_x)^\downarrow\rangle \tag{6.43}$$

$$= \langle x^\uparrow, \rho_C(x)^\downarrow\rangle + \langle x^\uparrow, \rho_C(z_x)^\downarrow - \rho_{\tilde{C}}(z_x)^\downarrow\rangle + \langle x^\uparrow, \rho_{\tilde{C}}(x)^\downarrow - \rho_C(x)^\downarrow\rangle \tag{6.44}$$

$$\geq \sigma_C(x) - (\|\rho_C(x)^\downarrow - \rho_{\tilde{C}}(x)^\downarrow\| + \|\rho_C(z_x)^\downarrow - \rho_{\tilde{C}}(z_x)^\downarrow\|)\|x\|, \tag{6.45}$$

where we used the fact that, $\forall y \in \Pi, \langle x^\uparrow, \rho_{\tilde{C}}(z_x)^\downarrow\rangle \geq \langle x^\uparrow, \rho_{\tilde{C}}(y)^\downarrow\rangle$, that $\langle x^\uparrow, \rho_C(x)^\downarrow\rangle = \langle x, \rho_C(x)\rangle = \sigma_C(x)$ by Theorem 4.4 and the Cauchy–Schwarz inequality and the fact that $\|x^{\uparrow}\| = \|x\|$. Finally, since for any $u, v \in \mathbb{R}^n$, $\|u^{\downarrow} - v^{\downarrow}\| \leq \|u - v\|$,

$$1 - \varrho_C(x, z_x) \leq \sigma_C(x)^{-1}\|x\|(\|\rho_C(x) - \rho_{\tilde{C}}(x)\| + \|\rho_C(z_x) - \rho_{\tilde{C}}(z_x)\|). \tag{6.46}$$

Given that for any $x \in \Pi^+$, $\|x\| \leq 1$ then for any $x \in \mathcal{R}_C$,

$$\|x - z_x\|^2 \leq 2\min\left(\sigma_C(x), \frac{\sigma_C(z_x)^2}{\sigma_C(x)}\right)\left(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|\right)^2 \tag{6.47}$$

$$\times (\|\rho_C(x) - \rho_{\tilde{C}}(x)\| + \|\rho_C(z_x) - \rho_{\tilde{C}}(z_x)\|). \tag{6.48}$$

It remains to express the distance between the spectra as a function of the distance between the matrices.

Step 4. For any $x \in \Pi^+$, we have

$$\|\rho_C(x) - \rho_{\tilde{C}}(x)\| \leq \|\sigma_C^{-1}(x)Cx - \sigma_C^{-1}(x)\tilde{C}x\| + \|\sigma_C^{-1}(x)\tilde{C}x - \sigma_{\tilde{C}}^{-1}(x)\tilde{C}x\| \tag{6.49}$$

$$\leq \sigma_C^{-1}(x)\|C - \tilde{C}\|\|x\| + |\sigma_C^{-1}(x) - \sigma_{\tilde{C}}^{-1}(x)|\|\tilde{C}x\|, \tag{6.50}$$

where the second term can also be dominated by $\|C - \tilde{C}\|$ since

$$|\sigma_C^{-1}(x) - \sigma_{\tilde{C}}^{-1}(x)|\|\tilde{C}x\| \leq \sigma_C^{-1}(x)\sigma_{\tilde{C}}^{-1}(x)\frac{|\sigma_C^2(x) - \sigma_{\tilde{C}}^2(x)|}{\sigma_C(x) + \sigma_{\tilde{C}}(x)}\|\tilde{C}^{\frac{1}{2}}\|\|\tilde{C}^{\frac{1}{2}}x\| \tag{6.51}$$

$$= \sigma_C^{-1}(x)\frac{|\langle(C - \tilde{C})(x), x\rangle|}{\sigma_C(x) + \sigma_{\tilde{C}}(x)}\|\tilde{C}^{\frac{1}{2}}\| \tag{6.52}$$

$$\leq \sigma_C^{-1}(x)\|C - \tilde{C}\|\|\tilde{C}^{\frac{1}{2}}\|\|x\|^2(\sigma_{\tilde{C}}(x) + \sigma_C(x))^{-1}. \tag{6.53}$$

Then for any $x \in \mathcal{R}_C$,

$$\|x - z_x\|^2\left(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|\right)^{-2} \tag{6.54}$$

$$\leq 2\left(\sigma_C(x)\|\rho_C(x) - \rho_{\tilde{C}}(x)\| + \frac{\sigma_C(z_x)^2}{\sigma_C(x)}\|\rho_C(z_x) - \rho_{\tilde{C}}(z_x)\|\right) \tag{6.55}$$

$$\leq 2\|C - \tilde{C}\|\left(\left(1 + \|\tilde{C}^{\frac{1}{2}}\|(\sigma_{\tilde{C}}(x) + \sigma_C(x))^{-1}\right)\right. \tag{6.56}$$

$$\left. + \frac{\sigma_C(z_x)}{\sigma_C(x)}\left(1 + \|\tilde{C}^{\frac{1}{2}}\|(\sigma_{\tilde{C}}(z_x) + \sigma_C(z_x))^{-1}\right)\right) \tag{6.57}$$

$$\leq 2\|C - \tilde{C}\|\left(1 + \|\tilde{C}^{\frac{1}{2}}\|\sigma_C^{-1}(x) + \sigma_C^{-1}(x)\left(1 + \|\tilde{C}^{\frac{1}{2}}\|\right)\right), \tag{6.58}$$

which implies that for any $x \in \mathcal{R}_C$,

$$d(x, \mathcal{R}_{\tilde{C}})^2 \leq 2(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|)^2\left(1 + \mathrm{DR}(C) + 2\|\tilde{C}^{\frac{1}{2}}\|\mathrm{DR}(C)\right)\|C - \tilde{C}\| \tag{6.59}$$

$$\leq 4(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|)^2\mathrm{DR}(C)(1 + \|\tilde{C}^{\frac{1}{2}}\|)\|C - \tilde{C}\|. \tag{6.60}$$

In a similar way, for any $x \in \mathcal{R}_C$ one has

$$\frac{\sigma_C(x - z_x)^2}{2\|C - \tilde{C}\|} \leq (\|C^{-\frac{1}{2}}\mathbf{1}\|^2 \sigma_C(x)\sigma_C(z_x) + 1) \tag{6.61}$$

$$\times \left(\frac{\sigma_C(z_x)}{\sigma_C(x)}\left(1 + \frac{\|\tilde{C}^{\frac{1}{2}}\|}{\sigma_{\tilde{C}}(x) + \sigma_C(x)}\right) + 1 + \frac{\|\tilde{C}^{\frac{1}{2}}\|}{\sigma_{\tilde{C}}(z_x) + \sigma_C(z_x)}\right), \tag{6.62}$$

which implies

$$\min_{\tilde{x} \in \mathcal{R}_{\tilde{C}}} \sigma_C(x - \tilde{x})^2 \leq 2(1 + \|C^{-\frac{1}{2}}\mathbf{1}\|^2)(1 + \mathrm{DR}(C))(1 + \|\tilde{C}^{\frac{1}{2}}\|\mathrm{DR}(C))\|C - \tilde{C}\|. \tag{6.63}$$

Similarly, for any $x \in \mathcal{R}_C$,

$$1 - \varrho_C(x, z_x) \leq \frac{\|x\|}{\sigma_C(x)}\left(\frac{\|x\|}{\sigma_C(x)}\left(1 + \frac{\|\tilde{C}^{\frac{1}{2}}\|\|x\|}{\sigma_{\tilde{C}}(x) + \sigma_C(x)}\right)\right. \tag{6.64}$$

$$\left. + \frac{\|z_x\|}{\sigma_C(z_x)}\left(1 + \frac{\|\tilde{C}^{\frac{1}{2}}\|\|z_x\|}{\sigma_{\tilde{C}}(z_x) + \sigma_C(z_x)}\right)\right)\|C - \tilde{C}\|, \tag{6.65}$$

and thus,

$$\min_{\tilde{x} \in \mathcal{R}_{\tilde{C}}} (1 - \varrho_C(x, \tilde{x})) \leq \frac{2\left(1 + \mu_{\max}(\tilde{C})^{\frac{1}{2}}(\mu^+_{\min}(\tilde{C})^{\frac{1}{2}} + \mu^+_{\min}(C)^{\frac{1}{2}})^{-1}\right)}{\mu^+_{\min}(C)}\|C - \tilde{C}\|. \tag{6.66}$$

Step 5. Putting everything together, for any $w \in \mathcal{R}_\Sigma$,

$$d(w, \mathcal{R}_{\tilde{\Sigma}}) \leq (1 + \max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|)\Big(\max(\sigma)\|\sigma^{-1}\|\|\tilde{\sigma}^{-1}\|\|\sigma - \tilde{\sigma}\| \tag{6.67}$$

$$+ 2\max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|(\|C^{-\frac{1}{2}}\| + \|C^{-\frac{1}{2}}\mathbf{1}\|)\left(\mathrm{DR}(C)(1 + \|\tilde{C}^{\frac{1}{2}}\|)\right)^{\frac{1}{2}}\|C - \tilde{C}\|^{\frac{1}{2}}\Big). \tag{6.68}$$

Therefore, the Hausdorff distance satisfies

$$d_{\mathcal{H}}(\mathcal{R}_\Sigma, \mathcal{R}_{\tilde{\Sigma}}) = \max\left(\max_{w \in \mathcal{R}_\Sigma} d(w, \mathcal{R}_{\tilde{\Sigma}}), \max_{\tilde{w} \in \mathcal{R}_{\tilde{\Sigma}}} d(\tilde{w}, \mathcal{R}_\Sigma)\right) \tag{6.69}$$

$$\leq \max(c_1(\sigma, \tilde{\sigma}), c_1(\tilde{\sigma}, \sigma))\|\sigma - \tilde{\sigma}\| + \max(c_2(\Sigma, \tilde{\Sigma}), c_2(\tilde{\Sigma}, \Sigma))\|C - \tilde{C}\|^{\frac{1}{2}}, \tag{6.70}$$

where

$$c_1(\sigma, \tilde{\sigma}) := \max(\sigma)\|\sigma^{-1}\|(1 + \max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|)\|\tilde{\sigma}^{-1}\|, \tag{6.71}$$

$$c_2(\Sigma, \tilde{\Sigma}) := 2(1 + \max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|)\max(\tilde{\sigma})\|\tilde{\sigma}^{-1}\|(\|C^{-\frac{1}{2}}\| + \mathrm{DR}_C(C^{-1}\mathbf{1})) \tag{6.72}$$

$$\times (\mathrm{DR}(C)(1 + \|\tilde{C}^{\frac{1}{2}}\|))^{\frac{1}{2}}. \tag{6.73}$$

Then, given that we have (4.36) and that $\mathrm{DR}_C(C^{-1}\mathbf{1}) = \langle C^{-1}\mathbf{1}, \mathbf{1}\rangle^{\frac{1}{2}} \geq 0$, then $\mathrm{DR}(C) \leq \mathrm{DR}_C(C^{-1}\mathbf{1}) = \langle C^{-1}\mathbf{1}, \mathbf{1}\rangle^{\frac{1}{2}} \leq \sqrt{n/\mu_m}$ and we obtain the announced bounds. In the same way, (6.60) [respectively, (6.66)] implies the bound we announced on $d_{\mathcal{H}}(\mathcal{R}_C, \mathcal{R}_{\tilde{C}})$ [respectively, $\tilde{d}_{\mathcal{H}}(\mathcal{R}_C, \mathcal{R}_{\tilde{C}})$]. □

The previous proposition allows us to characterize the stability of the set of maximally ρ-presentative portfolios, which we comment on further. As per our discussion before the proposition, we may assume that asset volatilities are fixed. Suppose now that we are given a correlation matrix C. We show that for any matrix $\tilde{C}$ that converges to C, the distance between their associated maximally ρ-presentative sets converges to zero along with the volume of their symmetric difference.

Beyond the volume of the difference between maximally ρ-presentative sets, the notions of proximity between portfolios we considered are: the Euclidean distance between their weights, their tracking error and their correlation. It is shown furthermore that given any two matrices C and $\tilde{C}$, the constants exhibited in the proposition allow to control how far from each other their associated maximally ρ-presentative sets will be for these latter three measures. In particular, if we choose any compact neighborhood K that contains C, there will be constants associated to K, such that any matrix $\tilde{C}$ belonging to K that converges to C does so at a rate that is bounded by a fixed constant that does not depend on $\tilde{C}$.

However, the constants that control the convergence rates are of a different nature. While the Hausdorff distance associated to the Euclidean distance is essentially controlled by the analog of a condition number μ_M/μ_m, the Hausdorff distance induced by the correlation distance $d_\varrho : (x, y) \mapsto \sqrt{2(1 - \varrho(x, y))}$ is controlled by its long-only version μ_M/μ_m^+ which is the analog of the *long-only condition number* $\mu_{\max}/\mu_{\min}^+$. The latter involves the smallest eigenvalue of a matrix over the positive orthant and improves over the usual condition number since $\mu_{\min} \leq \mu_{\min}^+$.

To illustrate the importance of a long-only condition number in practice, we consider a situation where the number of assets n increases and we compare the stability of the usual condition number with that of its long-only counterpart. For this purpose, we perform a numerical experiment where we consider the daily time series for 587 stocks of the MSCI USA (having discarded those that did not trade at least 90% of the days over March 2016–March 2017). We sort them by decreasing capitalization and form successively 587 universes with increasing n. For each one of these universes, we compute the condition number of the sample correlation matrix C and the inverse of $\mathrm{DR}(C)$ that results from a quadratic optimization. Since $n^{-1}\mu_{\min}(C) \leq \mathrm{DR}(C)^{-2} \leq \mu_{\min}^+(C)$, we then obtain an upper bound for the long-only condition number. The left inequality was proven in the end of the previous proof. Let us now prove the rightmost inequality:

$$\mathrm{DR}(C)^{-1} = \min_{x \in \Pi^+} \sigma_C(x) = \min_{\substack{x \succeq 0 \\ x \neq 0}} \frac{\langle Cx, x\rangle^{\frac{1}{2}}}{\|x\|_1} \leq \min_{\substack{x \succeq 0 \\ x \neq 0}} \frac{\langle Cx, x\rangle^{\frac{1}{2}}}{\|x\|_2} = \sqrt{\mu_{\min}^+(C)}. \tag{6.74}$$

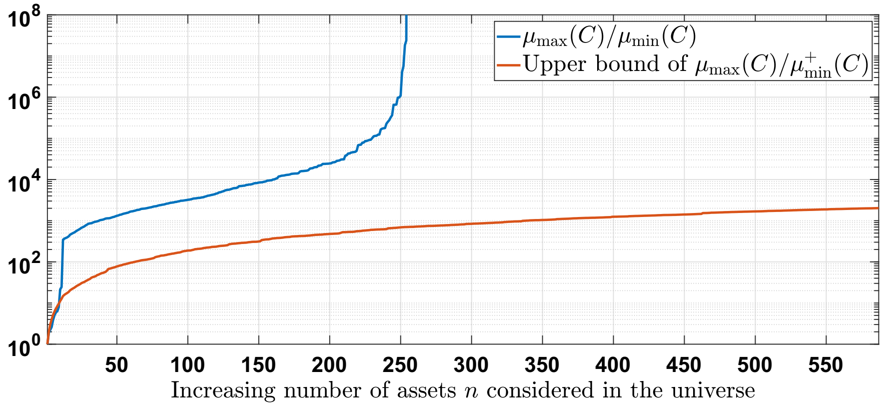

Fig. 5. Usual condition number versus long-only condition number for an increasing number of assets n. The upper bound on the long-only condition number indicates that it remains finite even if n is larger than the sample size.

We plot both condition numbers as a function of n in Fig. 5, which shows that, as opposed to the usual condition number, the long-only condition number remains finite even for a number of assets that is larger than the sample size. This is consistent with the fact that some long-only portfolio construction processes may remain stable in terms of d_ϱ even for problems involving ill-conditioned covariance matrices.

We will leave for future research the improvement of the bounds provided in Proposition 6.1, let alone the proof that they are optimal. For practical use, such bounds could also be significantly improved by considering a set of correlation matrices that reflect typical financial markets structures (e.g. with a dominant eigenvalue and associated "market" eigenvector). Finally, we note that even if we considered distances between sets of portfolios, the stability results we obtained could be transposed to some of the long-only portfolios we study in Sec. 5.1, with the concept of "long-only" eigenvalue and condition number playing a central role.

Remark 6.2.

(i) Let $d(A,B) := \lambda_{n-1}(A \,\Delta\, B)$, the pseudometric that we consider in the previous proposition. In Groemer (2000), the author proves that for two convex bodies $A, B \subset \mathbb{R}^n$ one may write $d(A,B) \leq c_{A,B}\, d_{\mathcal{H}}(A,B)$ and $d_{\mathcal{H}}(A,B) \leq C_{A,B}\, d(A,B)$ where $c_{A,B}$ and $C_{A,B}$ depend on A, B and n.

However, in our problem this result cannot be used directly as the set of maximally ρ-presentative portfolios is not necessarily convex. We could however rely on it and argue as in the proof of statement (i) of the proposition to show that $\lim_{\tilde{\Sigma}\to\Sigma}, d_{\mathcal{H}}(\mathcal{R}_\Sigma, \mathcal{R}_{\tilde{\Sigma}}) = 0$. This is a result that we obtain in an alternative way since it is implied by the local Hölder condition.

Furthermore, to get such a regularity property using the approach used in the proof of statement (i) of the previous proposition, we could have relied on the results of Schymura (2014) where the author gives an upper bound to the volume of the symmetric difference of a bounded set $A \subset \mathbb{R}^n$ and a perturbation $g(A)$. This bound involves the length of the boundary of A as measured by the Hausdorff measure. Unfortunately, the result was established only for a transform g that is a composition of a translation and a rotation and the paper does not consider the action of a positive-definite linear operator such as Σ.

(ii) Combining (6.35), (6.36) and Proposition 2.5 we obtain

$$\forall w, \tilde{w} \in \Pi^+, |\mathrm{DR}(w) - \mathrm{DR}(\tilde{w})| \leq \mathrm{DR}(\bar{w}) d_\varrho(w, \tilde{w}). \tag{6.75}$$

This inequality shows that two portfolios that are increasingly correlated to each other will have a Diversification Ratio that is increasingly similar. Conversely, this inequality may also be useful to obtain upper bounds on the potential correlation of portfolios that are far apart using their Diversification Ratios only.

7. Applications

7.1. *The Core Properties of the constrained MDP*

This section is dedicated to a theoretical application of Proposition 5.7 that also uses some elements of the proof of Proposition 5.8. We state two equivalent definitions of the constrained MDP — as defined in Sec. 5.2 — that extend to the constrained case the *First and Second Core Properties* of Choueifaty *et al.* (2013).

Proposition 7.1 (First Core Property). *The MDP w_r^* with volatility-adjusted maximum weight $1/r$ satisfies the following properties*:

(i) *The correlation of the portfolio to any asset that is held is smaller or equal to the correlation between the portfolio and any asset that is not held.*
(ii) *The correlation of the portfolio to any asset that saturates the max constraint is smaller or equal to the correlation between the portfolio and any asset that does not saturate the constraint.*
(iii) *The correlation of the portfolio to any asset that is held and does not saturate the max constraint is constant.*

Conversely, any portfolio in $\Pi^+_{\sigma,r}$ that satisfies (i), (ii) *and* (iii) *is necessarily the constrained MDP w_r^*.*

Proof. Statement (i) reads $(w_r^*)_i = 0$ and $(w_r^*)_j > 0 \implies (\rho(w_r^*))_i \geq (\rho(w_r^*))_j$. It is enough to prove the results for $x^* = \phi^{-1}(w_r^*)$. Employing the KKT theorem as in the proof of Proposition 5.8 and using the same notations, $(Cx^*)_i = s + (\lambda_i - \mu_i) \geq s \geq s + (\lambda_j - \mu_j) = (Cx^*)_j$. In a similar way, we get (ii). Claim (iii) follows readily from (Case 2) in the same proof, i.e. $\lambda_j = \mu_j = \lambda_i = \mu_i = 0$, hence $(Cx^*)_i = (Cx^*)_j$.

Conversely, assume that $w = \phi(x)$ satisfies all claims. This implies that

$$0 = x_i < x_j \leq x_k = r^{-1} \implies (Cx)_i \geq (Cx)_j \geq (Cx)_k. \tag{7.1}$$

If m is the number of saturated stocks, $m \leq r$ since $\langle x, \mathbf{1} \rangle = 1$. Let I (respectively, J) be the set of indices of the stocks that saturate (respectively, do not saturate) the constraint, then

$$\sigma_C(x)^2 = \sum_{i \in I}(Cx)_i x_i + \sum_{j \in J}(Cx)_j x_j = \frac{1}{r}\sum_{i=1}^{m}(Cx)_{(i)} + \sum_{j \in J}(Cx)_j x_j \tag{7.2}$$

by (i) and (ii). Now given that by (iii), $\exists \nu \in \mathbb{R}/\forall j \in J, (Cx)_j = \nu$, it follows

$$\sigma_C(x)^2 - \frac{1}{r}\sum_{i=1}^{m}(Cx)_{(i)} = \nu \sum_{j \in J} x_j = \nu\left(1 - \sum_{i \in I} x_i\right) = \nu\left(1 - \frac{m}{r}\right), \tag{7.3}$$

where in the two last identities we used the fact that $\langle x, \mathbf{1} \rangle = 1$ and the definition of I. However,

$$\frac{1}{r}\sum_{i=1}^{r}(Cx)_{(i)} = \frac{1}{r}\sum_{i=1}^{m}(Cx)_{(i)} + \frac{1}{r}\sum_{j=m+1}^{r}(Cx)_{(j)} = \frac{1}{r}\sum_{i=1}^{m}(Cx)_{(i)} + \frac{\nu}{r}(r-m) \tag{7.4}$$

by (ii) and (iii). Thus, $\frac{1}{r}\sum_{i=1}^{r}(Cx)_{(i)} = \sigma_C(x)^2$ which concludes the proof by Proposition 5.7. □

Proposition 7.2 (Second Core Property). *The following statements are equivalent*:

(i) w_r^* *is the MDP with volatility-adjusted maximum weight constraint* $1/r$,
(ii) $w_r^* \in \Pi_{\sigma,r}^+$ *is such that for any* $w_r \in \Pi_{\sigma,r}^+$, $\mathrm{DR}(w_r) \leq \varrho(w_r, w_r^*)\mathrm{DR}(w_r^*)$.

Proof. (ii) $\implies$ (i) as $\forall w_r \in \Pi_{\sigma,r}^+$, $\mathrm{DR}(w_r) \leq \varrho(w_r, w_r^*)\mathrm{DR}(w_r^*) \leq \mathrm{DR}(w_r^*)$, i.e. w_r^* has the highest DR that can be achieved over the set of constraints $\Pi_{\sigma,r}^+$. (i) $\implies$ (ii) is simply the inequality (5.18). □

7.2. *A not-so-typical saddle-point problem*

In Proposition 5.2, we introduced the problem

$$\max_{w \in \mathbb{R}^n \setminus \{0\}} \min_{\theta \in \Pi^+} \varrho(w, \theta),$$

which may remind us of a minimax matrix game problem or *saddle-point problem* but it is different in nature as $(w, \theta) \mapsto \varrho(w, \theta)$ is not (even quasi) concave–convex. Let us pick an example with three assets with covariance

$$\Sigma = \begin{pmatrix} 1.0 & -0.3 & -0.4 \\ -0.3 & 1.0 & -0.5 \\ -0.4 & -0.5 & 1.0 \end{pmatrix}. \tag{7.5}$$

In this case, the MDP and MV are both equal to $w^* \approx (0.31, 0.32, 0.36)$. This example is an opportunity to show the differences between the objective functions

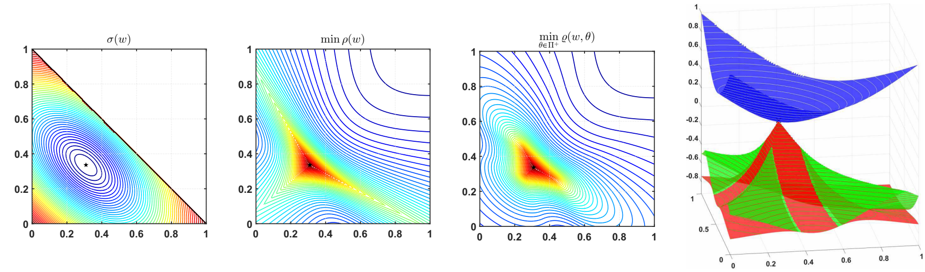

Fig. 6. As indicated by the level lines, the MV/MDP depicted by a star minimizes $w \mapsto \sigma(w)$ while maximizing both the nonsmooth $w \mapsto \min\rho(w)$ and $w \mapsto \min_{\theta\in\Pi^+}\varrho(w,\theta)$ as proven in Proposition 5.2. The nonconvex superlevels indicate that the latter functions are not quasi-concave. From the two charts in the middle, it is clear that there are $w \in \Pi^+$ with $\min_{\theta\in\Pi^+}\varrho(w,\theta) < \min\rho(w)$. By Lemma 3.4, these portfolios are such that $\rho(w) \not\geq 0$. To the right, we plot at the top the graph of $w \mapsto \sigma(w)$ and below the graphs of $w \mapsto \min(\rho(w))$ and $w \mapsto \min_{\theta\in\Pi^+}\varrho(w,\theta)$ which are not smooth. We can observe that these three graphs intersect at a single point that is the MV/MDP.

of Proposition 5.2 where we proved

$$\max_{w\in\mathbb{R}^n\setminus\{0\}}\min\rho(w) = \max_{w\in\mathbb{R}^n\setminus\{0\}}\min_{\theta\in\Pi^+}\varrho(w,\theta) = \min_{w\in\Pi^+}\sigma(w). \tag{7.6}$$

These identities may remind us a primal–dual framework with the *primal* and *dual* problems on both ends.

One can argue as in the proof of Proposition 5.4 to reduce the search set to those long-short w that sum to one. Therefore, to illustrate these problems, we depict in Fig. 6 the level lines of the objective functions $(x,y) \mapsto \sigma(x,y,1-x-y)$ and $(x,y) \mapsto \min\rho(x,y,1-x-y)$. As they are significantly different we also draw the level lines of $(x,y) \mapsto \min_{\theta\in\Pi^+}\varrho((x,y,1-x-y),\theta)$. This latter chart allows us to better understand the second identity of (5.2) in Proposition 5.2 which is implied by Lemma 3.4 and to shed some light on the remark that follows this lemma (see the caption of Fig. 6 for the details). To further illustrate the duality suggested by the inequalities (5.16) we depict the graphs of the three functions we just mentioned.

7.3. *Realized max ρ-presentativity and realized diversification*

Let us recall that Proposition 4.5 asserts that any maximally ρ-presentative w satisfies the necessary condition

$$\mathrm{DR}(w) \geq \mathrm{DR}(w_{\mathrm{evw}})/\varrho(w, w_{\mathrm{evw}}). \tag{7.7}$$

In this subsection, we show how this bound could be used to identify funds that qualify for being maximally ρ-presentative without knowing their composition. Indeed, assuming that a fund is constantly rebalanced to maintain w, $\varrho(w, w_{\mathrm{evw}})$ can be measured by simply computing the correlation between the time series of w and that of w_{evw}. The latter is computed thanks to the series of the assets of the universe. Similarly, using time series only, the realized DR of a portfolio with unknown composition can be also measured thanks to the following result already proved in (4.36).

Proposition 7.3. *Let* $\bar{w} := \Sigma^{-1}\sigma/\|\Sigma^{-1}\sigma\|_1$ *be the portfolio that maximizes* DR *over* Π, *then for any* $w \in \Pi$,

$$\mathrm{DR}(w) = \mathrm{DR}(\bar{w})\varrho(\bar{w}, w). \tag{7.8}$$

Thus the long-only MDP is the portfolio that is most correlated to the long-short MDP amongst all long-only portfolios. In this sense, the long-only MDP is the *projection* of the long-short MDP over long-only portfolios. Note that, using this identity, one can reformulate (5.18) in a way that may remind us a triangle inequality:

$$\forall w_r \in \Pi^+_{\sigma,r}, \quad \varrho(w_r, \bar{w}) \leq \varrho(w_r, w_r^*)\varrho(w_r^*, \bar{w}). \tag{7.9}$$

Here, we used $\varrho(w_r^*, \bar{w}) \geq 0$ which follows from Proposition 7.3.

Let us get back to our idea, and perform a numerical experiment where we consider daily time series for 464 stocks of the MSCI USA (having discarded those that did not trade at least 90% of the days over January 2013–March 2017). Using the Bloomberg Fund Screening module, we similarly considered daily time series for the funds that satisfy the following:

```
Market Status: Active;                       Fund Strategy: Blend;
Fund Asset Class Focus: Equity;              Fund Primary Share Class: Yes;
Fund Geographical Focus: International;      First Date: <= 1/1/2013;
Currency: USD;                               Fund Total Assets (mil): >100M.
Fund Pricing Frequency: Daily;
```

We discarded 71 funds that had obvious price synchronization issues, ending up with 2278 funds for a total of \$7500 billion, i.e. about 80% of the total net assets invested in equity funds in the USA in Q1/2016 (according to the International Investment Funds Association report of June 28, 2016). Note that the computation of $\mathrm{DR}(w)$ involves the inversion of the sample covariance Σ which is possible with probability 1 since we have *circa* 1300 daily returns for each one of the 464 stocks. In Fig. 7, we depict the realized $\mathrm{DR}(w)$ of these funds as a function of $\mathrm{DR}(w_{\mathrm{evw}})/\varrho(w, w_{\mathrm{evw}})$ and indicate the identity function using a dashed line. A live fund depicted by a red star satisfies the necessary condition as it lies above the dashed line, as do the forward-looking constrained MDPs that are indeed maximally ρ-presentative by Proposition 5.7.

On a different topic, the fact that some funds have a DR that is less than one may indicate that they are not long-only or composed of assets that are outside of the considered universe. Indeed, as we do not have access to their compositions, we cannot guarantee that they are invested solely in the MSCI USA selection.

Finally, we refer to Appendix A.2 for some additional illustrations of the theoretical results of this paper that are based on this dataset of funds.

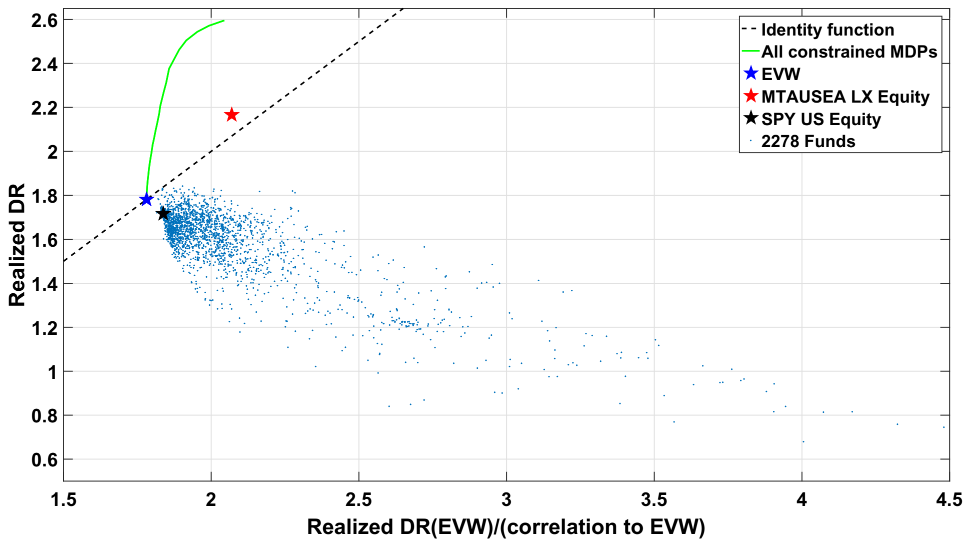

Fig. 7. Realized DR and $\mathrm{DR}(w_{\mathrm{evw}})/\varrho(w_{\mathrm{evw}}, \cdot)$ in the USA from January 2013 to March 2017 for 2278 funds representing 80% of the total net assets invested in equity funds in the USA in Q1/2016. The fund depicted with a blue star is a theoretical and forward-looking EVW portfolio. The green curve depicts all the forward-looking constrained MDPs. The green curve and the dashed line — that depicts the identity — meet precisely at the forward-looking EVW portfolio. The fund in red is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR whereas the fund in black replicates the S&P500 index. The blue dots depict all other funds. Only the portfolios that are above the dashed line qualify for being maximally ρ-presentative as they satisfy the necessary condition $\mathrm{DR}(w) \geq \mathrm{DR}(w_{\mathrm{evw}})/\varrho(w, w_{\mathrm{evw}})$ of Proposition 4.5. The green curve corresponds to portfolios that are indeed maximally ρ-presentative by Proposition 5.7.

8. Conclusions

As an alternative to portfolio weights w, we have introduced the equivalent representation offered by the correlation spectrum $\rho(w)$, i.e. the vector of correlations of a portfolio to all the assets of an investment universe. This new representation naturally leads to the notion of ρ-presentative portfolio — such as the ERC, MV and MDP — which allows an investor to be positively exposed to all assets without necessarily being invested in all of them.

An important contribution of the paper is the concept of maximally ρ-presentative portfolio, which maximizes its aggregated exposure to all assets. The real-valued function f that measures the aggregated exposure $f(\rho(w))$ of the portfolio is assumed to be symmetric, concave and increasing. We have first proved that maximally ρ-presentative portfolios are long-only using a key lemma: for any portfolio that is not long-only, there always exists a long-only portfolio that is more correlated to all assets. A characterization of this new class of portfolios is then provided: its members are essentially the long-only portfolios whose exposures form a nonincreasing function of their volatility-adjusted weights. In particular, this implies that these portfolios are diversified. Using the structure imposed by this characterization, we have also proven that these portfolios are rare amongst long-only ones. However, well-known members of this class of portfolios include the EVW, ERC and MDPs that can be constrained with maximum weights. Furthermore, we have shown that the set of maximally ρ-presentative portfolios satisfies an original geometric property, namely it is the union of a finite number of polytopes. Thanks to this particular structure and the above-mentioned characterization, we have shown that this set satisfies a local Hölder regularity property, i.e. the distance between two covariance matrices controls locally the Hausdorff distance between the corresponding maximally ρ-presentative sets. In particular, in this result, we used the concept of long-only eigenvalue which seems particularly relevant for characterizing the stability of long-only optimized portfolios.

As we have seen, the aggregation function f provides a fairly general trade-off between the average and the dispersion of the exposures of a portfolio. Using Schur concave and increasing functions offers an avenue for further research to generalize the classic mean-variance approach to portfolio construction.

Having tackled the no-short sales constraints, we have studied the impact of adding maximum weight constraints to the MDP and MV. The results provided in this paper extend the analytical results of Jagannathan and Ma (2003), as their impact on the MV objective is made explicit and known *a priori*. Furthermore, in a context where the covariance matrix has to be estimated, this yields a plain interpretation of the impact of these constraints on the objective: reducing its estimation error. We leave for further research the formal study of the biases and estimation variance reduction induced by the addition of constraints on the MDP and MV. It should be noted that even in a setting where returns are Gaussian, the problem is challenging as it depends on the order statistics of $\rho_{\hat{\Sigma}}(w^*(\hat{\Sigma}))$.

On another topic, many of the arguments in this paper (KKT, ball compactness, continuity of convex functions, etc.) rely on the fact that the analysis was performed in a finite-dimensional setting. It would be interesting to extend these results to a setting where there is a continuum of assets. In particular, we wonder what the set of maximally ρ-presentative portfolios would be in this case.

Also, the characterization of these rare portfolios that is at the core of this paper satisfies a purely algebraic property that deserves a more thorough analysis. Moreover, this problem seems to share connections with for instance task scheduling problems where the idle time is minimized under all permutations of the tasks.

Finally, our results are general as they only rely on the positivity of the covariance matrix and provide a unifying framework for many well-known alternative investment strategies, that are shown to maximize under no constraints their overall exposure to all assets. Furthermore, beyond their financial implications, the findings of this article may be useful in other fields where correlations are used to measure interactions.

Appendix A.

A.1. *A general composition formula for the spectrum of a portfolio of portfolios*

Proposition A.1. *If we have $m > 1$ portfolios $w_i \in \Pi$ that we arrange in columns in an $n \times m$ matrix W and a portfolio of portfolios $\theta \in \mathbb{R}^m$ with $\theta \succeq 0$ and $\langle \theta, \mathbf{1} \rangle = 1$ such that $\sigma(W\theta) \neq 0$, then*

$$\rho(W\theta) = d(W\theta)\rho(W)\Phi_{\sigma(w_i)_i}(\theta), \tag{A.1}$$

where

(i) $d(W\theta) = \frac{\langle \theta, (\sigma(w_i))_i \rangle}{\sigma(W\theta)} \in [1, +\infty)$,
(ii) $\rho(W)$ *is the $n \times m$ matrix whose columns are $\rho(w_i)$,*
(iii) $\Phi_{\sigma(w_i)_i}(\theta) = \frac{\theta \odot (\sigma(w_i))_i}{\langle \theta, (\sigma(w_i))_i \rangle} \in \mathbb{R}^m$ *and has nonnegative components that sum to one.*

If we take a k-homogeneous ($k > 0$) and concave $f : \mathbb{R}^n \to \mathbb{R}$ that we apply to the columns of $\rho(W)$ we have a property similar to strict convexity:

$$f \circ \rho(W\theta) \geq d(W\theta)^k [f \circ \rho(W)]\Phi_{\sigma(w_i)_i}(\theta). \tag{A.2}$$

If for any $w \in \Pi$, $\theta \in \Pi^+$, we consider the function $f(x) = \langle x, \frac{w}{\sigma(w)} \rangle$, $m := n$ and $W = Id$, then

$$\varrho(w, \theta) = \mathrm{DR}(\theta)\langle \rho(w), \phi(\theta) \rangle. \tag{A.3}$$

The latter proposition generalizes Proposition 2.4 and relates it to identity (3.4) in Lemma 3.4.

A.2. *Realized* $\mathbf{RM}_r$

Let us observe that as for the realized DR, the realized RM_r of a portfolio (introduced in Sec. 5.2.1) may be measured without knowing its composition as the realized $\rho(w)_i$ is simply the correlation between the time series of w with that of asset i. Therefore, we can perform another numerical experiment by placing ourselves in the same setting as in Sec. 7.3 and considering the same universe of 464 stocks and 2278 funds. For each fund w and every integer $r \leq 464$, one can compute $\mathrm{RM}_r(w)$ using the sample correlation. In Fig. A.1, we plot all the curves $r \mapsto \mathrm{RM}_r(w)$ (that are nondecreasing by definition of RM).

For $r \leq 32$, the fund maximizing $\mathrm{RM}_r(w)$ amongst all funds is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR. Observe that $r = 32$ is the smallest integer such that this fund has not the highest $\mathrm{RM}_r(w)$ suggesting that its *implicit volatility-adjusted maximum weight constraint* is larger than 3.13%.

By Proposition 7.3, we can also plot in Fig. A.2 the realized DR of all 2278 funds as a function of their RM_{32}. In addition, we plot in green the constrained MDPs computed over the whole window for all values of r. These are all forward-looking

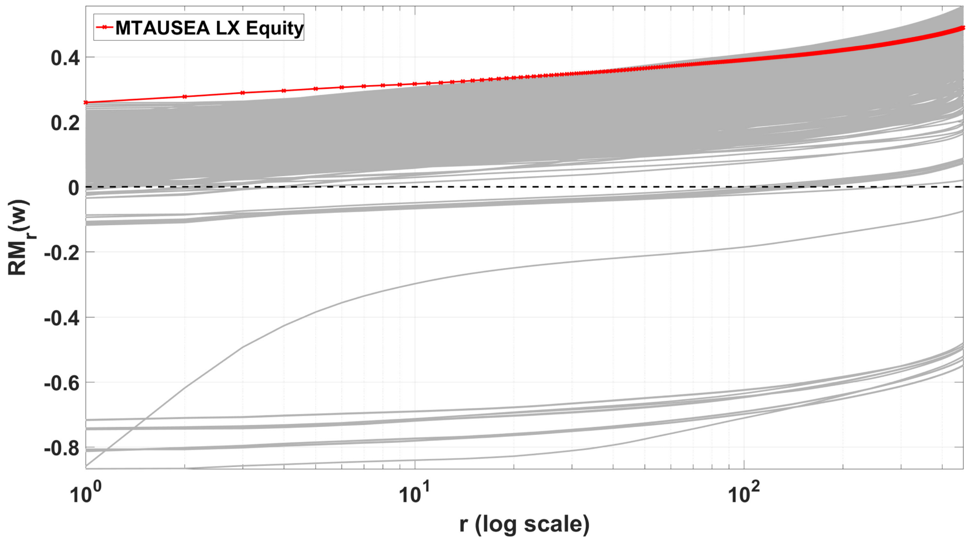

Fig. A.1. $(\mathrm{RM}_r)_r$ in the USA for 2278 funds from January 2013 to March 2017. The red fund aims to maximize the DR.

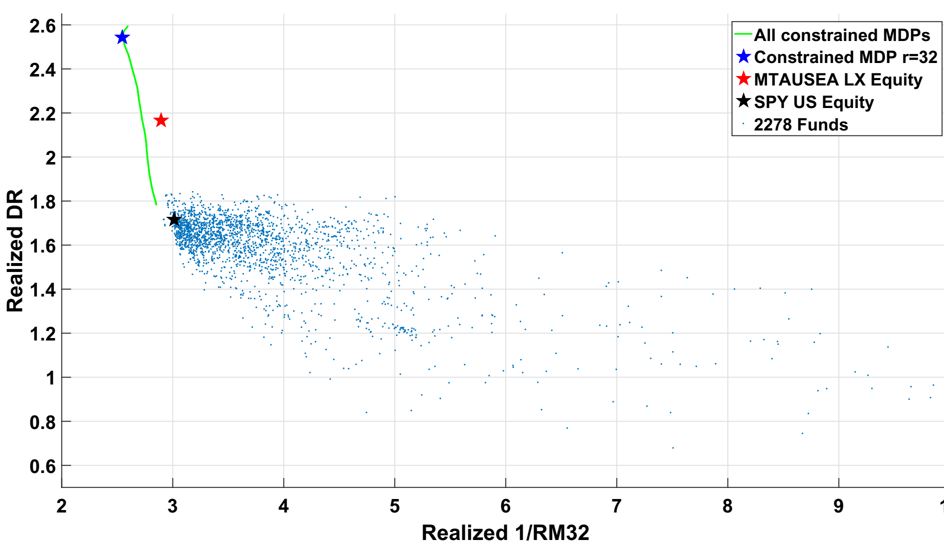

Fig. A.2. Realized DR and $1/\mathrm{RM}_{32}$ in the USA from January 2013 to March 2017 for 2278 funds representing about 80% of the total net assets invested in equity funds in the USA in Q1/2016. The blue star is a theoretical and forward-looking constrained MDP with r=32. The green curve depicts all forward-looking constrained MDPs. The fund in red is the `MOST DIV TOBAM A/B US EQ-A` that targets the highest investable DR whereas the one in black replicates the S&P500 index. The blue dots depict all other funds.

portfolios. Observe that the unconstrained MDP is suboptimal in terms of RM_{32} whereas $\mathrm{DR}(w^*_{32}) = \mathrm{RM}_{32}(w^*_{32})^{-1}$ as proven in Proposition 5.7. Similarly, using for instance Proposition 3.5, we isolate in the list below the funds that may not be long-only. Their names clearly indicate that they are indeed all "short" or "bear" funds:

```
ADVISORSHARES RANGER EQ BEAR;
DIREXION DAILY FINL BEAR 3X;
DIREXION DAILY S&P 500 BEAR;
DIREXION DLY SM CAP BEAR 3X;
GRIZZLY SHORT FUND;
PROSH ULTRAPRO SHORT S&P 500;
PROSHARES SHORT DOW30;
PROSHARES SHORT QQQ;
PROSHARES SHORT RUSSELL2000;
PROSHARES SHORT S&P500;
PROSHARES ULTPRO SHRT DOW30;
PROSHARES ULTRAPRO SHORT QQQ;
PROSHARES ULTRAPRO SHRT R2K;
PROSHARES ULTRASHORT DOW30;
PROSHARES ULTRASHORT QQQ;
PROSHARES ULTRASHORT R2000;
PROSHARES ULTRASHORT S&P500.
```

One could have also used inequality (5.18) or the second assertion of Lemma 4.3 to isolate funds that may not be long-only and thus not maximally ρ-presentative. Finally, in Fig. A.1, we can also identify funds that are not ρ-presentative as they have negative RM_1.

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